PDEs (Huge Area, only scratching the surface)

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Types of PDEs

hyperbolic, *parabolic*, and *elliptic* depending on their *characteristics*, or curves of information propagation; examples:

Hyperbolic $$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$$ 1-D wave equation

IVPs, Cauchy problems; start at t=0 and evolve the solution

Parabolic $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right)$$ diffusion equation

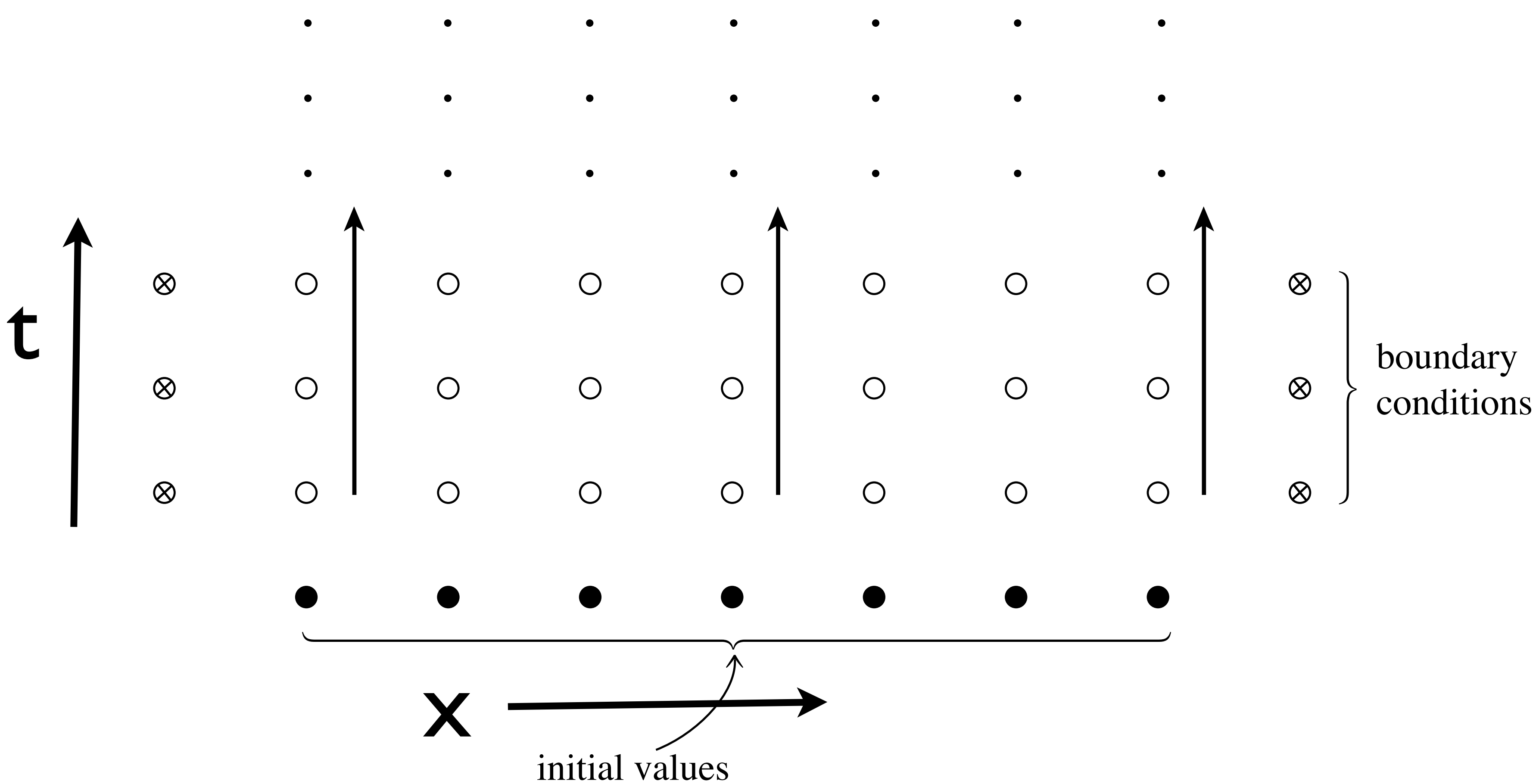
Elliptic $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x, y)$$ Poisson equation

boundary value problem; no evolution! find the solution which satisfies PDE and boundary conditions

simple prototypes are very instructive and guide the solution of more complicated problems

IVPs

main concern is stability
accuracy while being fast
remember CFL condition!



needed! specified in a physical way

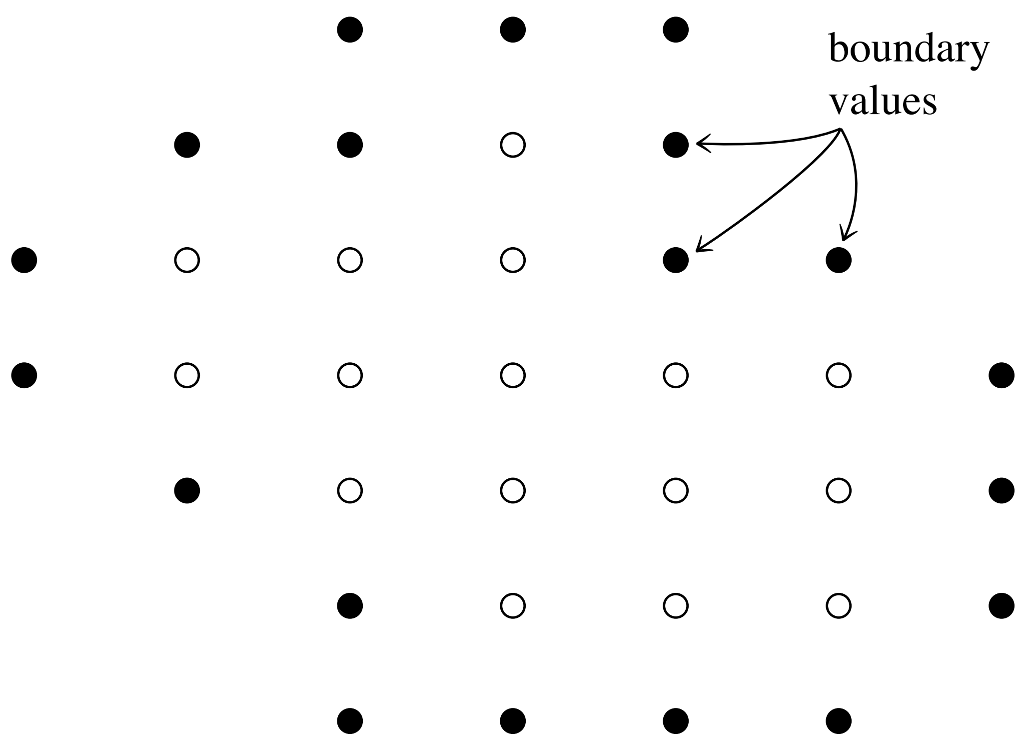
dependent variables to be propagated forward in time f

evolution equation, e.g., advection eq. $\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0$

prefer to have $\frac{\partial}{\partial t}$ on LHS

boundary conditions: Dirichlet BCs (specify boundary values of f at all times), Neumann (specify derivatives at boundary); & combinations specified in physically plausible way

BVPs



FD reduces to: Ax=b

main concern: *efficiency* of algorithms
methods generally stable
recall Jacobi relaxation, conjugate-gradient, steepest descent

recall: importance of spectral radius, condition number!

- What are the variables (dependent & independent)?
- What equations are satisfied in the interior of the region of interest? $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x, y)$$
- What equations are satisfied by points on the boundary of the region of interest? (Here Dirichlet and Neumann conditions are possible choices for elliptic second-order equations, but more complicated boundary conditions can also be encountered.)

all conditions on BVP must be satisfied "simultaneously,"
boil down to solution of large numbers of simultaneous algebraic equations. When such equations are nonlinear, they are usually solved by linearization and iteration;
solution of special, large linear sets of equations (thus matrix methods are important for BVPs!)

Flux-Conservative IVPs

flux-conservative equation

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{\partial \mathbf{F}(\mathbf{u})}{\partial x}$$

a lot of PDEs in physics are conservation laws of this form

conserved flux

$$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x} \qquad u = f(x - vt)$$

our old friend: *advection equation*

FTCS is unstable! recall VNSA, linear stability analysis by expanding in Fourier modes (WKB limit) can apply even for nonlinear eq. by linearizing; not rigorous but practically useful

Lax/Lax-Friedrichs Method in FTCS in time derivative term $u_j^n \rightarrow \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right)$

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{v\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

stable if CFL condition $\frac{|v|\Delta t}{\Delta x} \leq 1$

modified equation: $$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2\Delta t}\nabla^2 u$$

stabilizing diffusion term *numerical dissipation*, or *numerical viscosity*

Lax Method

for resolved modes kΔx<<1, amplification factor agrees with analytic result
small scale modes not accurately captured but:
unstable for FTCS & stable/damped for Lax => while Lax is fine can't use FTCS

recall that stability concerns are supreme for IVPs

Lax method in 2 variables, e.g., $\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial r}{\partial t} = v\frac{\partial s}{\partial x} \qquad r \equiv v\frac{\partial u}{\partial x}$$

$$\frac{\partial s}{\partial t} = v\frac{\partial r}{\partial x} \qquad s \equiv \frac{\partial u}{\partial t}$$

$$\frac{\partial}{\partial t}\begin{bmatrix} r \\ s \end{bmatrix} = \frac{\partial}{\partial x}\begin{bmatrix} vs \\ vr \end{bmatrix}$$

Lax in 2D

$$r_j^{n+1} = \frac{1}{2}(r_{j+1}^n + r_{j-1}^n) + \frac{v\Delta t}{2\Delta x}(s_{j+1}^n - s_{j-1}^n)$$

$$s_j^{n+1} = \frac{1}{2}(s_{j+1}^n + s_{j-1}^n) + \frac{v\Delta t}{2\Delta x}(r_{j+1}^n - r_{j-1}^n)$$

VNSA in 2 variables:

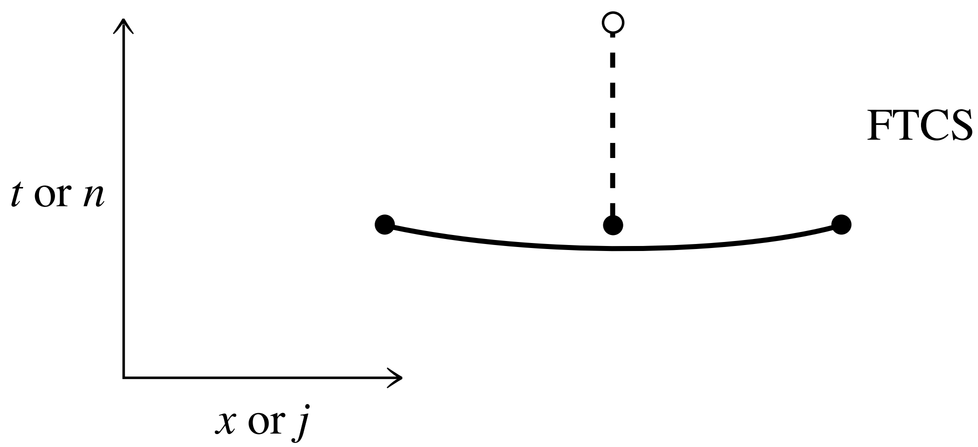
$$\begin{bmatrix} r_j^n \\ s_j^n \end{bmatrix} = \xi^n e^{ikj\Delta x}\begin{bmatrix} r^0 \\ s^0 \end{bmatrix}$$

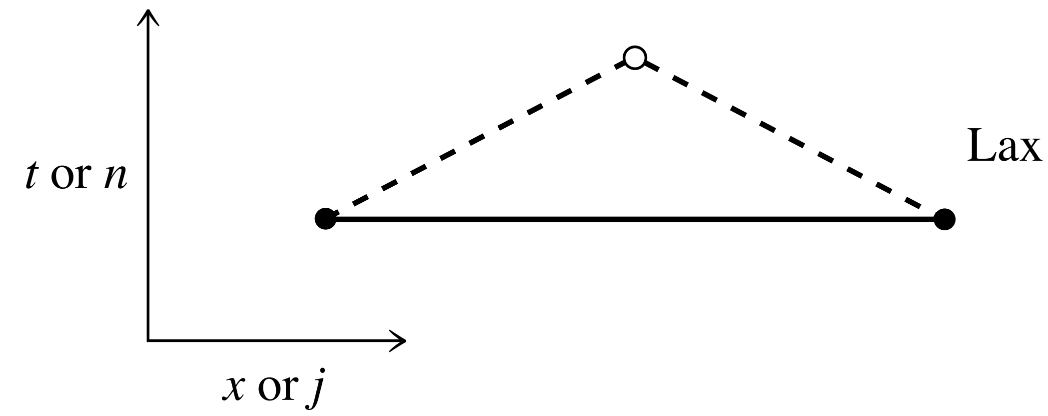
$$\begin{bmatrix} (\cos k\Delta x) - \xi & i\frac{v\Delta t}{\Delta x}\sin k\Delta x \\ i\frac{v\Delta t}{\Delta x}\sin k\Delta x & (\cos k\Delta x) - \xi \end{bmatrix} \cdot \begin{bmatrix} r^0 \\ s^0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

multiple eigenvalues instead of single amplification factor

$$\xi = \cos k\Delta x \pm i\frac{v\Delta t}{\Delta x}\sin k\Delta x$$ again stable if $$\frac{|v|\Delta t}{\Delta x} \le 1$$

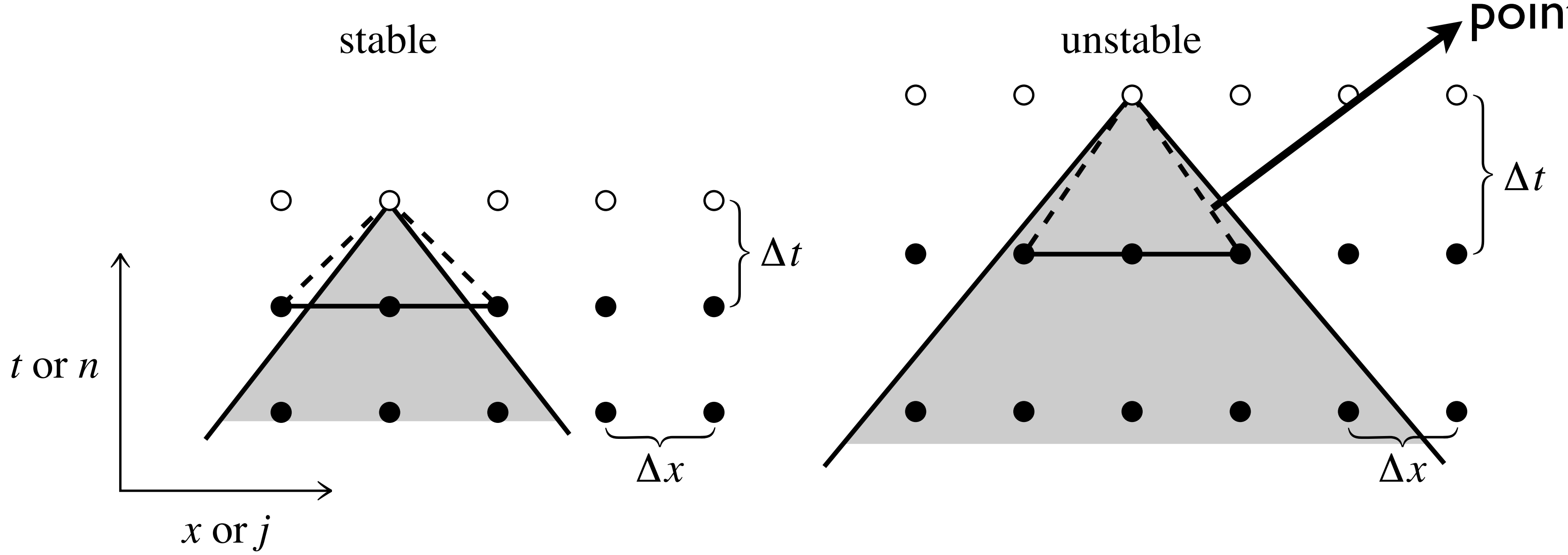
x-t diagrams & characteristics





CFL condition in x-t diagram

information beyond nearest points needed=>instability



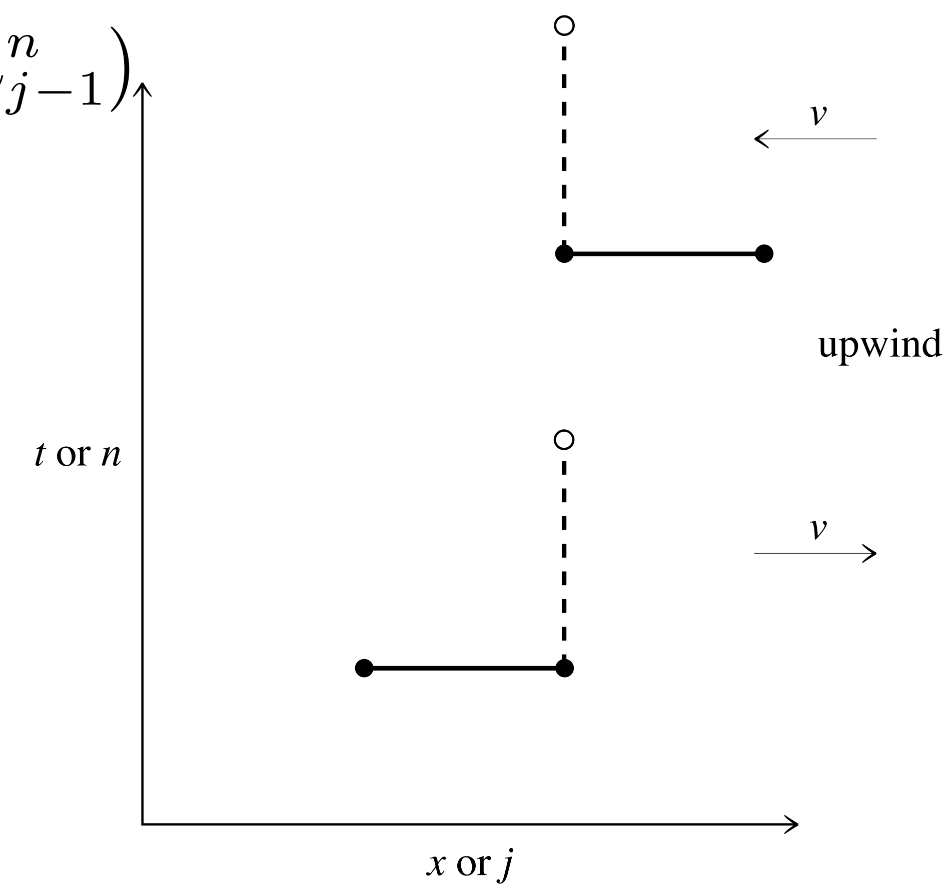
shaded: finite domain of dependence for hyperbolic eqs.
CFL condition: characteristics should not cross more than a grid cell

Upwinding

$$u = f(x - vt)$$ solution should only be affected by points in *upwind* direction

Lax Method $$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{v\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

symmetric w.r.t. +/- direction, unlike advection eq.; leading order error: O(Δt, Δx²/Δt)



Upwind Scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v_j^n \begin{cases} \dfrac{u_j^n - u_{j-1}^n}{\Delta x}, & v_j^n > 0 \\ \dfrac{u_{j+1}^n - u_j^n}{\Delta x}, & v_j^n < 0 \end{cases}$$

better than Lax, even if first order accurate in space! since inherits the upwind property of original equations

stability again governed by CFL condition: $\frac{|v|\Delta t}{\Delta x} \leq 1$

Δx accuracy improved by using a larger stencil in upwind direction:

$$u_x^- = \frac{3u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2\Delta x}$$

O(Δt^2) accuracy in time

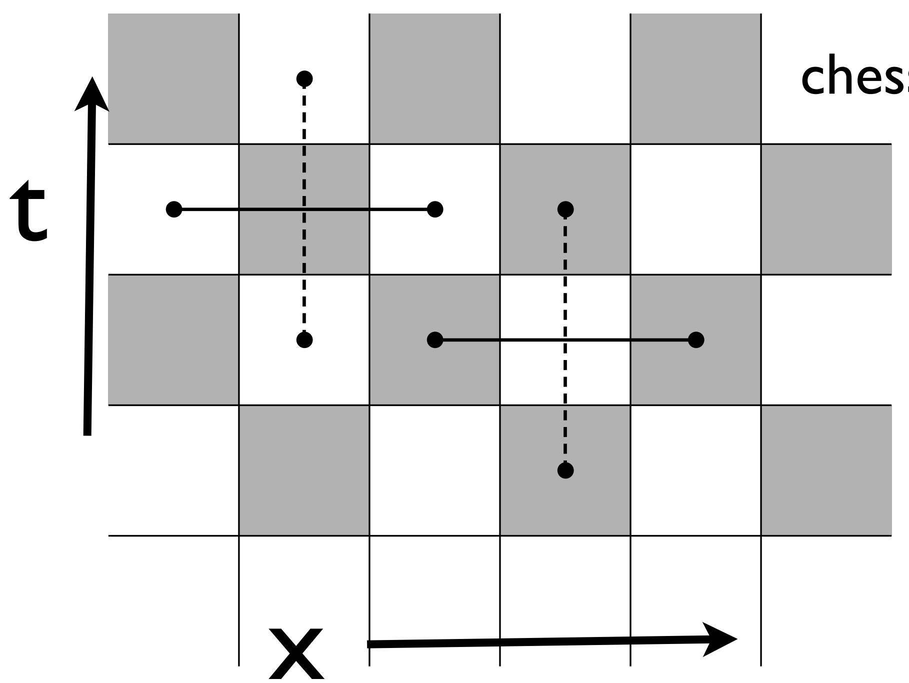
if method is O(Δt) but O(Δx^2) $\Delta t << v\Delta x$ must be chosen for accuracy
better to have same order of accuracy in space and time

staggered leapfrog **method**

$$u_j^{n+1} - u_j^{n-1} = -\frac{\Delta t}{\Delta x}(F_{j+1}^n - F_{j-1}^n) = -\frac{v\Delta t}{\Delta x}(u_{j+1}^n - u_{j-1}^n)$$

$$\xi = -i\frac{v\Delta t}{\Delta x}\sin k\Delta x \pm \sqrt{1 - \left(\frac{v\Delta t}{\Delta x}\sin k\Delta x\right)^2}$$

amplification factor; |r| = 1 for $\frac{|v|\Delta t}{\Delta x} \leq 1$



chess-board coupling

unstable for steep gradients!

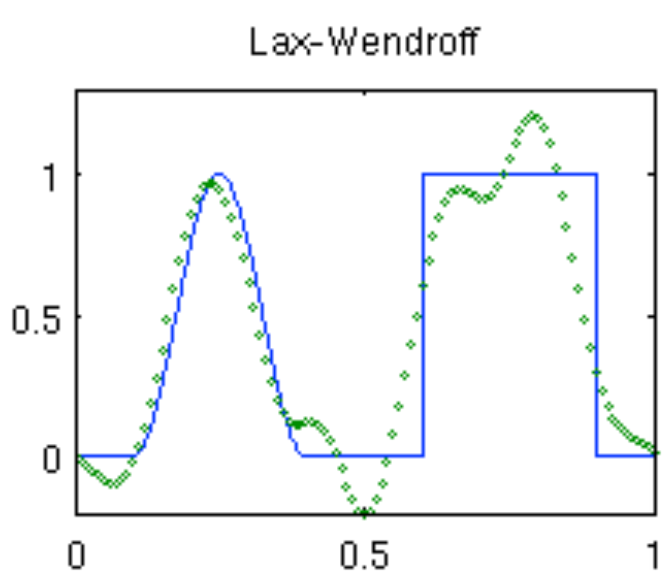
cured by adding small diffusion coefficient

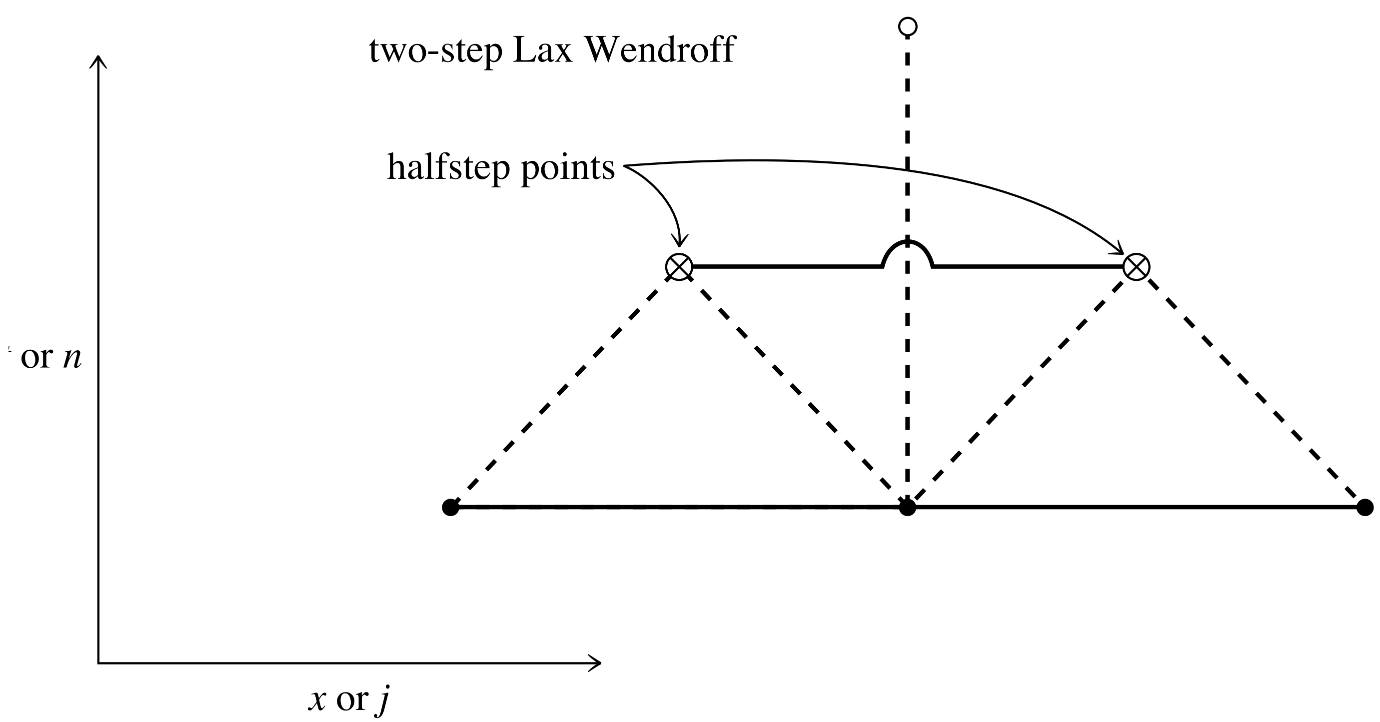
Lax-Wendroff

Lax step: $$u_{j+1/2}^{n+1/2} = \frac{1}{2}(u_{j+1}^n + u_j^n) - \frac{\Delta t}{2\Delta x}(F_{j+1}^n - F_j^n)$$

second order differencing in x,t $$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^{n+1/2} - F_{j-1/2}^{n+1/2}\right)$$

stable for $\frac{|v|\Delta t}{\Delta x} \le 1$ as seen in HW2





shock-capturing

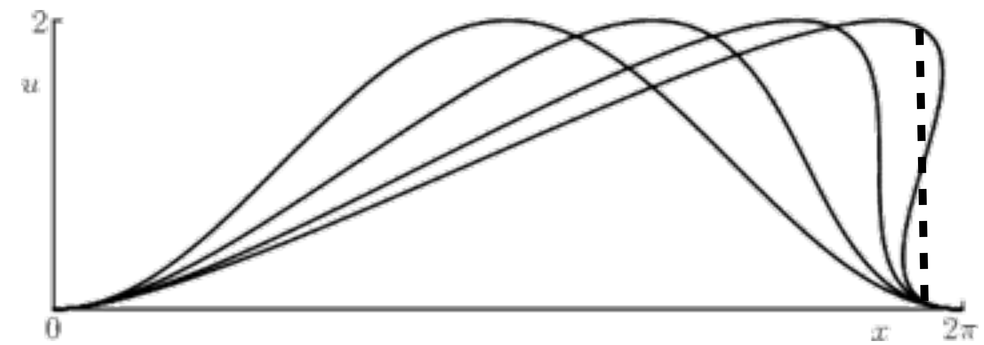
many eqs. Euler eqs., Burger's eq., etc. admit discontinuous solutions (a.k.a. shocks)
integral (weak) form of eqs. is suited for numerical solution
numerical methods should be able to capture shocks without Gibbs oscillations

artificial viscosity: numerical methods should be able to capture shocks without Gibbs oscillations

Burger's equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$

wave steepening

$$u_t + uu_x = \varepsilon u_{xx}$$



$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial q}{\partial x}$$

can't be multi-valued!

von Neumann Richtmyer artificial viscosity active only at shocks, l (2Δx): length over which shock is smoothened

$$q = \begin{cases} l^2\rho(\partial v/\partial x)^2 & \text{if} \quad (\partial v/\partial x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

monotonicity preserving/TVD, higher-order schemes; slope/flux limiters on *transport* terms

Godunov finite-volume schemes: solution described in terms of volume averaged quantities in cells
discontinuities at cell boundaries resolved into shocks/rarefactions and the Riemann problem solved exactly/approximately; huge area!

Diffusive IVPs

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right) \qquad F = -D\frac{\partial u}{\partial x}$$

D>0 for being well-posed

explicit FTCS for D=constant: $$\frac{u_j^{n+1} - u_j^n}{\Delta t} = D\left[\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}\right]$$

VNSA: $$\xi = 1 - \frac{4D\Delta t}{(\Delta x)^2}\sin^2\left(\frac{k\Delta x}{2}\right)$$

stability criterion: $$\frac{2D\Delta t}{(\Delta x)^2} \le 1$$

time-step ~ diffusion timescale across a cell

diffusion time-step: $$\tau \sim \frac{\lambda^2}{D}$$

$\lambda^2/(\Delta x)^2$ timesteps before anything happens at scales of interest

implicit/semi-implicit:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = D\left[\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2}\right]$$

fully implicit, unconditionally stable; HW2

$$\frac{\partial^2 u}{\partial x^2} = 0$$

equilibrium solution as $\Delta t \to \infty$ but only 1st order accurate in time

***Crank-Nicholson* scheme:** centered in space & time O(Δt^2,Δx^2)

$$\frac{u_j^{n+1}-u_j^n}{\Delta t}=\frac{D}{2}\left[\frac{(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1})+(u_{j+1}^n-2u_j^n+u_{j-1}^n)}{(\Delta x)^2}\right]$$

$$\xi=\frac{1-2\alpha\sin^2\left(\frac{k\Delta x}{2}\right)}{1+2\alpha\sin^2\left(\frac{k\Delta x}{2}\right)}$$

unconditionally stable, but as $\Delta t\to\infty$ $\left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1}=-\left(\frac{\partial^2 u}{\partial x^2}\right)^{n}$

=> initial disturbances at small scales are maintained at late times!

$D=D(x)$ $$\frac{u_j^{n+1}-u_j^n}{\Delta t}=\frac{D_{j+1/2}(u_{j+1}^n-u_j^n)-D_{j-1/2}(u_j^n-u_{j-1}^n)}{(\Delta x)^2}$$

$$D_{j+1/2}\equiv D(x_{j+1/2})$$

for stability: $\Delta t\le\min_j\left[\frac{(\Delta x)^2}{2D_{j+1/2}}\right]$

nonlinear $D=D(u)$ if $dz=D(u)du$ RHS= $\frac{z_{j+1}^{n+1}-2z_j^{n+1}+z_{j-1}^{n+1}}{(\Delta x)^2}$

now linearize $$z_j^{n+1}\equiv z(u_j^{n+1})=z(u_j^n)+(u_j^{n+1}-u_j^n)\left.\frac{\partial z}{\partial u}\right|_{j,n}$$
$$=z(u_j^n)+(u_j^{n+1}-u_j^n)D(u_j^n)$$

uncond. stable,
semi-implicit,
tridiagonal form

Multi-D IVPs

Lax in 2D: $$\frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{F} = -\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right)$$

$$x_j = x_0 + j\Delta$$
$$y_l = y_0 + l\Delta$$

$$u_{j,l}^{n+1} = \frac{1}{4}(u_{j+1,l}^n + u_{j-1,l}^n + u_{j,l+1}^n + u_{j,l-1}^n) - \frac{\Delta t}{2\Delta}(F_{j+1,l}^n - F_{j-1,l}^n + F_{j,l+1}^n - F_{j,l-1}^n)$$

VNSA for $F_x = v_x u,\quad F_y = v_y u \quad u_{j,l}^n = \xi^n e^{ik_x j\Delta} e^{ik_y l\Delta}$

$$\Delta t \le \frac{\Delta}{\sqrt{2}(v_x^2 + v_y^2)^{1/2}} \qquad \Delta t \le \frac{\Delta}{\sqrt{N}|v|}$$ in N dimensions; similarly for L-W

Diffusion Equation in Multidimensions

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Crank-Nicolson in 2D: $$u_{j,l}^{n+1} = u_{j,l}^n + \frac{1}{2}\alpha\left(\delta_x^2 u_{j,l}^{n+1} + \delta_x^2 u_{j,l}^n + \delta_y^2 u_{j,l}^{n+1} + \delta_y^2 u_{j,l}^n\right)$$

where $\alpha \equiv \frac{D\Delta t}{\Delta^2} \qquad \Delta \equiv \Delta x = \Delta y$

sparse but not easily solvable matrix:
Jacobi/related relaxation, CG

$$\delta_x^2 u_{j,l}^n \equiv u_{j+1,l}^n - 2u_{j,l}^n + u_{j-1,l}^n$$

for explicit methods in N-D $\Delta t \le \frac{\Delta^2}{2ND}$

Operator Splitting

alternating-direction implicit method (ADI)

results in easily solvable tridiagonal system

$$u_{j,l}^{n+1/2} = u_{j,l}^{n} + \frac{1}{2}\alpha\left(\delta_x^2 u_{j,l}^{n+1/2} + \delta_y^2 u_{j,l}^{n}\right)$$

$$u_{j,l}^{n+1} = u_{j,l}^{n+1/2} + \frac{1}{2}\alpha\left(\delta_x^2 u_{j,l}^{n+1/2} + \delta_y^2 u_{j,l}^{n+1}\right)$$

Operator Splitting Methods Generally

$$\frac{\partial u}{\partial t} = \mathcal{L}u \qquad \mathcal{L}u = \mathcal{L}_1 u + \mathcal{L}_2 u + \cdots + \mathcal{L}_m u$$

$$u^{n+(1/m)} = \mathcal{U}_1(u^n, \Delta t)$$

$$u^{n+(2/m)} = \mathcal{U}_2(u^{n+(1/m)}, \Delta t)$$

$$\cdots$$

$$u^{n+1} = \mathcal{U}_m(u^{n+(m-1)/m}, \Delta t)$$

$$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial x} + D\frac{\partial^2 u}{\partial x^2}$$ apply L-W for advection step & CN for diffusion

ADI: can think of as operator splitting

$$u^{n+1/m} = \mathcal{U}_1(u^n, \Delta t/m)$$

$$u^{n+2/m} = \mathcal{U}_2(u^{n+1/m}, \Delta t/m)$$

$$\cdots$$

$$u^{n+1} = \mathcal{U}_m(u^{n+(m-1)/m}, \Delta t/m)$$

very useful because we want to simulate several processes simultaneously!
several combinations of splitting: Strang splitting, etc.

Fourier Method for BVPs

$$\mathbf{A} \cdot \mathbf{u} = \mathbf{b}$$ for both linear and nonlinear (need to linearize in this case) problems

Fourier methods when regular boundaries & constant coefficients

2-D Poisson eq.:

$$\frac{u_{j+1,l} - 2u_{j,l} + u_{j-1,l}}{\Delta^2} + \frac{u_{j,l+1} - 2u_{j,l} + u_{j,l-1}}{\Delta^2} = \rho_{j,l}$$

w. different BCs

$$\mathbf{u}_{j-1} + \mathbf{T} \cdot \mathbf{u}_j + \mathbf{u}_{j+1} = \mathbf{g}_j \Delta^2$$

vector form at interior points

$$\mathbf{u}_j^T = (u_{j,0}, u_{j,1}, .., u_{j,L})$$

$$\mathbf{T} \cdot \mathbf{u}_j = u_{j,l+1} - 4u_{j,l} + u_{j,l-1}$$

discrete IFT

$$u_{jl} = \frac{1}{JL} \sum_{m=0}^{J-1} \sum_{n=0}^{L-1} \widehat{u}_{mn} e^{-2\pi i j m/J} e^{-2\pi i l n/L} \qquad \rho_{jl} = \frac{1}{JL} \sum_{m=0}^{J-1} \sum_{n=0}^{L-1} \widehat{\rho}_{mn} e^{-2\pi i j m/J} e^{-2\pi i l n/L}$$

$$\widehat{u}_{mn}\left(e^{2\pi im/J}+e^{-2\pi im/J}+e^{2\pi in/L}+e^{-2\pi in/L}-4\right)=\widehat{\rho}_{mn}\Delta^2 \qquad \widehat{u}_{mn}=\frac{\widehat{\rho}_{mn}\Delta^2}{2\left(\cos\frac{2\pi m}{J}+\cos\frac{2\pi n}{L}-2\right)}$$

discrete DFT $\widehat{\rho}_{mn}=\sum_{j=0}^{J-1}\sum_{l=0}^{L-1}\rho_{jl}\,e^{2\pi imj/J}e^{2\pi inl/L}$ compute u_{jl} by IFT

for periodic BCs $u_{jl}=u_{j+J,l}=u_{j,l+L}$

Dirichlet boundary condition $u=0$: use sin transforms which vanish at boundaries

$$u_{jl}=\frac{2}{J}\frac{2}{L}\sum_{m=1}^{J-1}\sum_{n=1}^{L-1}\widehat{u}_{mn}\sin\frac{\pi jm}{J}\sin\frac{\pi ln}{L} \qquad \widehat{\rho}_{mn}=\sum_{j=1}^{J-1}\sum_{l=1}^{L-1}\rho_{jl}\sin\frac{\pi jm}{J}\sin\frac{\pi ln}{L}$$

inhomogeneous BCs: for example $u = 0$ on all boundaries except $u = f(y)$ on the boundary $x = J\Delta$

u^H of the homogeneous equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ that satisfies the BC

$$u^H_{jl}=\frac{2}{L}\sum_{n=1}^{L-1}A_n\sinh\frac{\pi nj}{J}\sin\frac{\pi nl}{L} \quad \text{where} \quad A_n=\frac{1}{\sinh\pi n}\sum_{l=1}^{L-1}f_l\sin\frac{\pi nl}{L}$$

full solution: $u=u_{jl}+u^H_{jl}$

for general BCs $u=u'+u^B$ where $u'=0$ on boundary and $u^B=0$ everywhere *except* boundary; since boundary terms are known they can be taken to RHS; they affect RHS at the last active zone close to boundary; see NR for more

Cyclic Reduction

FFT methods are limited as applicable if PDE has constant coefficients; cyclic reduction is more general

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + b(y)\frac{\partial u}{\partial y} + c(y)u = g(x,y)$$

arises often for Poisson/Helmholz eq.
in cylindrical/spherical coordinates

this eq. can be FD as

$$\mathbf{u}_{j-1} + \mathbf{T}\cdot\mathbf{u}_j + \mathbf{u}_{j+1} = \mathbf{g}_j\Delta^2$$

O($N^2\log_2 N$) like FFT

$$\left.\begin{aligned}
&\mathbf{u}_{j-2} + \mathbf{T}\cdot\mathbf{u}_{j-1} + \mathbf{u}_j = \mathbf{g}_{j-1}\Delta^2\\
&\mathbf{u}_{j-1} + \mathbf{T}\cdot\mathbf{u}_j + \mathbf{u}_{j+1} = \mathbf{g}_j\Delta^2\\
&\mathbf{u}_j + \mathbf{T}\cdot\mathbf{u}_{j+1} + \mathbf{u}_{j+2} = \mathbf{g}_{j+1}\Delta^2
\end{aligned}\right\}\ \text{combine}$$

$$\mathbf{u}_{j-2} + \mathbf{T}^{(1)}\cdot\mathbf{u}_j + \mathbf{u}_{j+2} = \mathbf{g}_j^{(1)}\Delta^2$$

$$\mathbf{T}^{(1)} = 2\mathbf{1} - \mathbf{T}^2$$

$$\mathbf{g}_j^{(1)} = \Delta^2(\mathbf{g}_{j-1} - \mathbf{T}\cdot\mathbf{g}_j + \mathbf{g}_{j+1})$$

Taking the number of mesh points to be a power of 2

$$\underbrace{\mathbf{T}^{(f)}\cdot\mathbf{u}_{J/2}}_{\text{Tridiagonal system}} = \Delta^2\mathbf{g}_{J/2}^{(f)} \underbrace{- \mathbf{u}_0 - \mathbf{u}_J}_{\text{known BCs}}$$

two equations at level f – 1 involve $\mathbf{u}_{J/4}$ and $\mathbf{u}_{3J/4}$. The equation for $\mathbf{u}_{J/4}$ involves $\mathbf{u}_0$ and $\mathbf{u}_{J/2}$, both of which are known, and hence can be solved by the usual tridiagonal routine. A similar result holds true at every stage, so we end up solving J – 1 tridiagonal systems.

Relaxation methods

solve Ax=b iteratively; also think as a solution of time-dependent problem till it reaches steady state

$\mathcal{L}u = \rho$ elliptic equation in operator form written in diffusive form $\frac{\partial u}{\partial t} = \mathcal{L}u - \rho$

relaxes to the solution as t→∞

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \rho$$

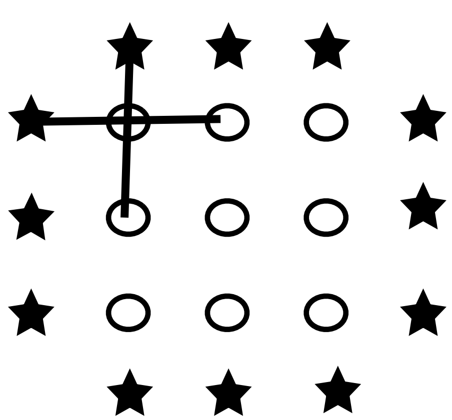
FTCS differencing: $u_{j,l}^{n+1} = u_{j,l}^{n} + \frac{\Delta t}{\Delta^2}\left(u_{j+1,l}^{n} + u_{j-1,l}^{n} + u_{j,l+1}^{n} + u_{j,l-1}^{n} - 4u_{j,l}^{n}\right) - \rho_{j,l}\Delta t$

choose $\Delta t = \Delta^2/4$. $u_{j,l}^{n+1} = \frac{1}{4}\left(u_{j+1,l}^{n} + u_{j-1,l}^{n} + u_{j,l+1}^{n} + u_{j,l-1}^{n}\right) - \frac{\Delta^2}{4}\rho_{j,l}$ iterate till conv.

same as *Jacobi's method* we saw earlier: Dx=-(L+U)x+b

Gauss-Seidel method: $u_{j,l}^{n+1} = \frac{1}{4}\left(u_{j+1,l}^{n} + u_{j-1,l}^{n+1} + u_{j,l+1}^{n} + u_{j,l-1}^{n+1}\right) - \frac{\Delta^2}{4}\rho_{j,l}$ use updated values

(L+D)x=-Ux+b



recall that speed of convergence is governed by spectral radius, which for Poisson eq. is

$\rho_s \simeq 1 - \frac{\pi^2}{2J^2}$ => number of steps to convergence∝J^2 (similar scaling for GS)

recall for CG: no. of steps for conv. $\sqrt{\kappa}$ ∝J

Successive Over-relaxation (SOR)

$$(\mathbf{L}+\mathbf{D})\cdot\mathbf{x}^{(r)} = -\mathbf{U}\cdot\mathbf{x}^{(r-1)} + \mathbf{b}$$ *Gauss-Seidel* method

$$\mathbf{x}^{(r)} = \mathbf{x}^{(r-1)} - (\mathbf{L}+\mathbf{D})^{-1}\cdot\underbrace{\left[(\mathbf{L}+\mathbf{D}+\mathbf{U})\cdot\mathbf{x}^{(r-1)} - \mathbf{b}\right]}_{-\mathbf{r}^{(r-1)}}$$

overcorrect

$$\mathbf{x}^{(r)} = \mathbf{x}^{(r-1)} + \omega(\mathbf{L}+\mathbf{D})^{-1}\cdot\mathbf{r}^{(r-1)}$$ remember $\varepsilon^r = x^r - x \to 0$ as $r^r = -A\varepsilon^r \to 0$

over-relaxation parameter

$$\epsilon^{(r)} = \left[\mathbf{I} - \omega(\mathbf{L}+\mathbf{D})^{-1}\mathbf{A}\right]\epsilon^{(r-1)}$$

- The method is convergent only for $0<\omega<2$. $0<\omega<1$ *under-relaxation*
- The *optimal* choice for ω is given by $\omega = \dfrac{2}{1+\sqrt{1-\rho_{\mathrm{Jacobi}}^2}}$ => $\rho_{\mathrm{SOR}} = \left(\dfrac{\rho_{\mathrm{Jacobi}}}{1+\sqrt{1-\rho_{\mathrm{Jacobi}}^2}}\right)^2$

for Poisson eq. $\omega \simeq \dfrac{2}{1+\pi/J}$

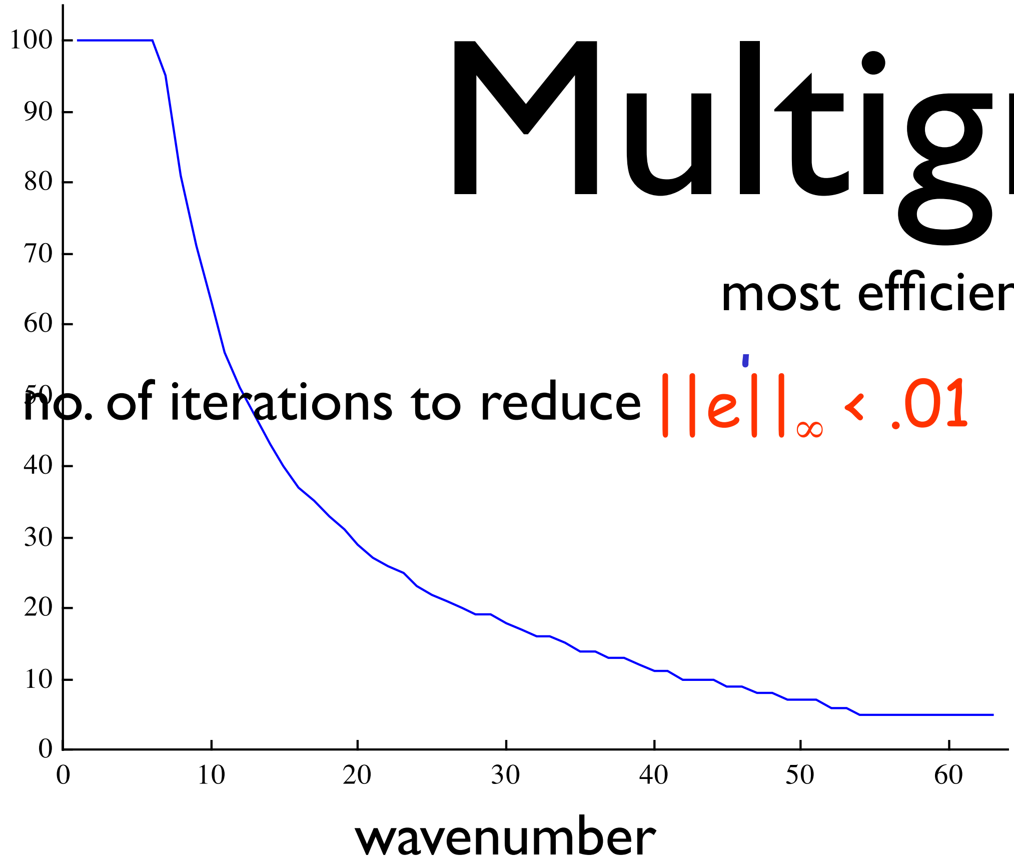
$$\rho_{\mathrm{SOR}} \simeq 1 - \frac{2\pi}{J}$$

=> convergence in steps ∝ J!

problem is that we need to know ω, not available in general

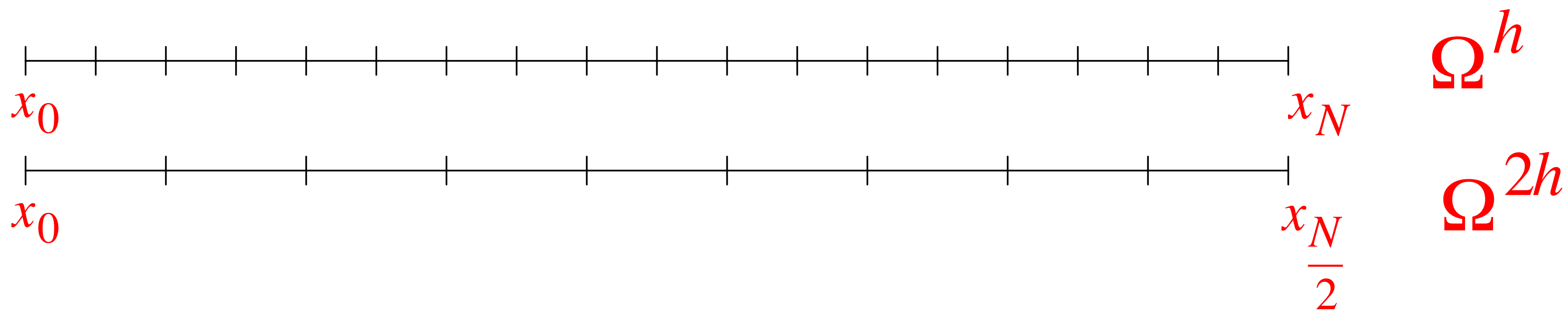
Multigrid methods

most efficient modern tool for well-behaved matrix eqs.!



Many relaxation schemes have the smoothing property, where oscillatory modes of the error are eliminated effectively, but smooth modes are damped very slowly.

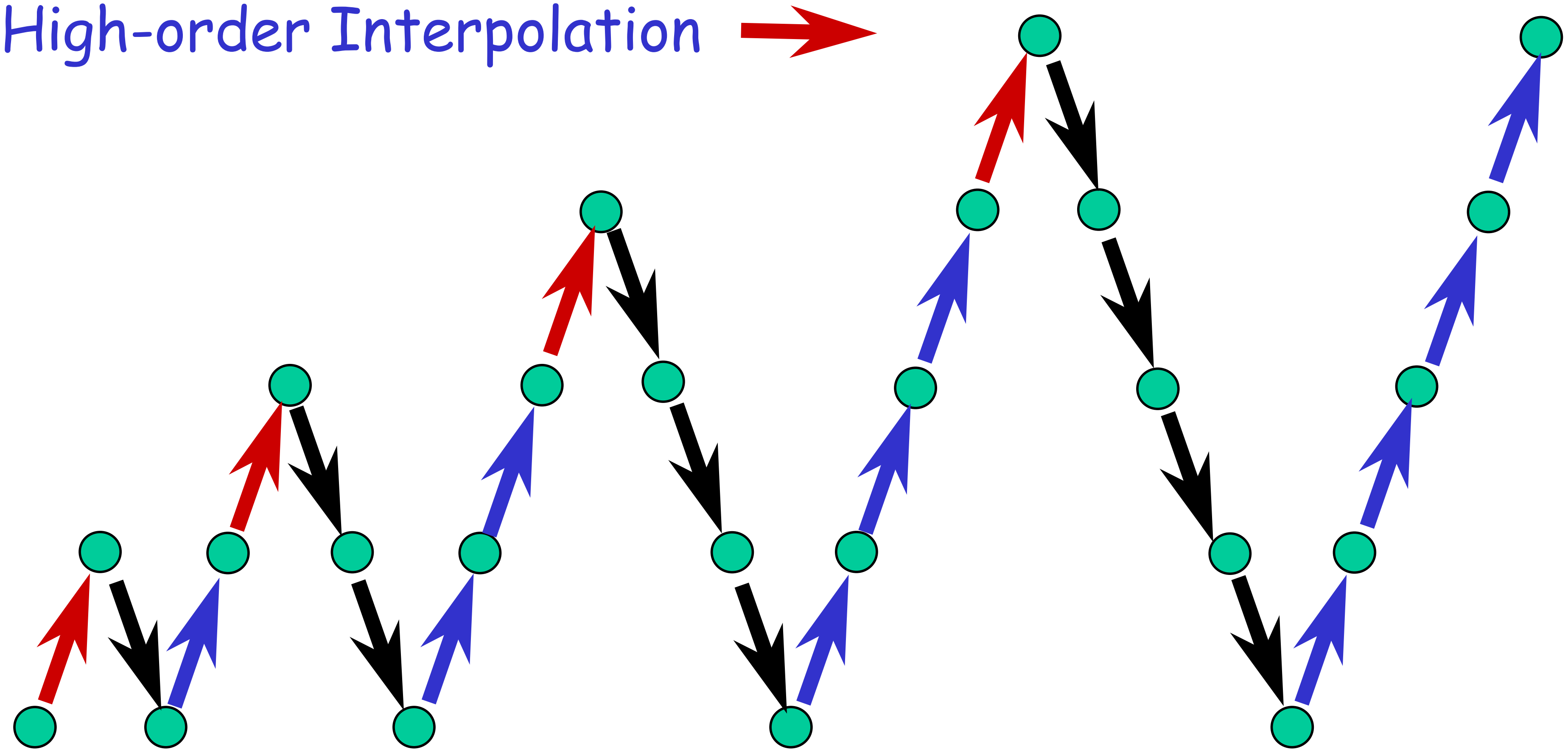
coarse grids cheaper better convergence



- Relax on Au=f on Ω^{4h} to obtain initial guess v^{2h}
- Relax on Au=f on Ω^{2h} to obtain initial guess v^{h}
- Relax on Au=f on Ω^{h} to obtain … final solution???

Full Multigrid (FMG)

- Restriction
- Interpolation
- High-order Interpolation



more in: A Multigrid Tutorial, Briggs et al.