Non-stationary Similarity Solutions and Power-law Tails in Stochastic Processes with Multiplicative Interactions

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Stochastic processes with asymmetrically multiplicative interactions are investigated analytically and numerically. In the model, two particles of positive quantities $x, y (> 0)$ interact and quantities $x, y$ convert into $x', y'$ as

$$
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = \begin{pmatrix}
c(1-a) & cb \\
d(a) & d(1-b)
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} (x \geq y)
$$

(1)

where $c, d \geq 0$ and $0 \leq a, b \leq 1$ are interaction parameters and two particles are selected randomly. Similarity solutions and power-law tails of the probability density function (PDF) of this model can be dealt with analytically in a similar way as in inelastic Maxwell models[1-3]. First, the Fourier transform of the master equation is performed. Secondly, moment equations are derived on the assumption that similarity solutions exist. Finally, transcendental equations which determine the growth rate $\gamma$ and the power-law exponent $s$ of the tail are obtained. The resulting equations of this model read:

$$
\gamma = \frac{1}{2} [c(1-a+b) + d(a+1-b)] + \frac{1}{2} (c-d)(1-a-b) A - 1,
$$

(2)

$$
A \equiv \int_0^\infty d\xi_2 \int_\xi_2^\infty d\xi_1 (\xi_1 - \xi_2) \Psi(\xi_1) \Psi(\xi_2),
$$

(3)

$$
\gamma s = \{c(1-a)^s + (da)^s - 1 \quad (s > 1)
$$

(4)

where $\Psi(\xi)$ is PDF rescaled by the scaling variable $\xi = xe^{-\gamma t}$. In the case $c = d$ or $a + b = 1$, the value of the growth rate $\gamma$ and power-law exponent $s$ can be obtained analytically because the equation for the first moment is closed. It should be emphasized that the power-law exponent $s$ is independent of the parameter $b$ when $c = d$. Transcendental equations are
formally same as those for symmetric interactions \( c=d, \ a=b \)[4]. However, the physical meaning of the parameters is completely different. In the other case \( c \neq d \) and \( a+b \neq 1 \), on the contrary, the value of \( \gamma \) is unable to be calculated analytically because the first moment equation is not closed. Then numerical simulations have been performed. Good agreement is achieved for both \( \gamma \) and \( s \). The effect of randomness in stochastic processes with symmetric interactions is also studied. In this case, \( c = d \equiv p + q \) and \( a = b \equiv p/(p+q) \) are random parameters with probability distribution \( \rho(p, q) \)[5]. The transcendental equations determining the growth rate \( \gamma \) and the power-law exponent \( s \) are

\[
\gamma = p + q - 1, \tag{5}
\]

\[
\gamma s = \bar{p} s + \bar{q} s - 1 \quad (s > 1) \tag{6}
\]

where bars denote average over \( \rho(p, q) \). Generally, randomness results in decrease of \( s \) and \( \gamma \). Explicit calculations are performed for two examples: uniformly distributed and two peaked systems. Significant influence is demonstrated when a bare growth rate is low and coupling is weak. It should be emphasized that even the sign of the growth rate can be changed from positive to negative growth. References