Ultra Cold Fermions: Dimensional Crossovers, Synthetic Gauge Fields and Synthetic Dimensions

A thesis
Submitted for the degree of
DOCTOR OF PHILOSOPHY
In the Faculty of Science

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July, 2016
Declaration

The work reported in this thesis is entirely original and has been performed by me at the Department of Physics, Indian Institute of Science, Bangalore, under the supervision of Prof. Vijay B. Shenoy. This thesis has not formed the basis for the award of any degree, diploma or similar title of any university or institution to the best of my knowledge.

Sudeep Kumar Ghosh,
July 20, 2016

SR No.: 9410-310-091-06860
This thesis is dedicated to my parents.
Acknowledgements

It has been a long and exciting journey of seven years and this thesis would not have been possible without the immense help of a lot of people over the years. First of all, I would like to acknowledge my thesis supervisor Prof. Vijay B. Shenoy. He has truly been a friend, philosopher and guide. To this date, his passion and dedication towards physics motivates me. Over the years I have committed many mistakes and I would like to thank him for tolerating me, being patient and sharing his wealth of knowledge with me.

I am equally grateful to my parents for their endless love and support throughout my graduate years and long before it. My father being a hardworking farmer taught me how much effort it needs to create something and how to value things in life. I am amazed by my mother’s passion towards cooking and feeding us and whenever she calls, she has the same question each and everyday: “What did you eat today?” I am also thankful to my brother, Binu, for many interesting conversations.

I am immensely grateful to my dearest friend cum girl-friend cum companion, Ria, for her unconditional love and continuous support throughout the years. I am forever in her debt for being there by my side and helping me during the ups and downs of my graduate years. I am also thankful to her sister, Raya, for flirting with me and having many enjoyable conversations.

I am thankful to all my course instructors: Prof. Vijay B. Shenoy, Prof. Subroto Mukerjee, Prof. H. R. Krishnamurthy, Prof. Diptiman Sen, Prof. Rahul Pandit, Prof. Sriram Ramaswamy, Prof. Vasant Natarajan, Prof. Chandan Dasgupta,
Prof. V. Venkatraman and all other teachers, for exposing me to the frontier of physics. I am also thankful to Prof. Manish Jain for enlightening discussions.

In addition, there is almost an endless list of friends and fellow graduate students who made these years enjoyable and productive. Firstly, I would like to acknowledge all my Integrated Ph.D. batch mates; Kallol, Shouvik, Sandip, Aamir, Vaisakh, Sainadh, Nagendra, Johnshon, Moupiya, Anindita and Maheswar; for our eventful days during course work. I am also thankful to my seniors: Amit Majhi, Sumilan, Debarghya, Rupamanjari, Saroj, Shaon, Chanchal, Nandan, Swarup, Sujit, Aveek, Debyan, Nirmalendu, Thakur, Rahul, Ketan, Bidya, Nitin Chandra, Subhamoy, Ranjan, Naveen Jingade, Ananyo Maitra, Kingshuk and others; and to my juniors: Rituparno, Gopal, Soumavo, Sayani, Kallol Sen and others for all the physics and non-academic discussions. I am also thankful to my other friends: Bidisha, Arvind, Samyaday, Arpan, Kapildeb and others.

I would like to thank my labmates Umesh, Jaynatha, Yogeshwar, Arijit, Abhaas, Adhip, Sambuddha, Amal for thought provoking discussions and technical helps. A special thanks to Adhip for making me realize how bad a PJ can be and for the manuscript corrections; and to Umesh as well for struggling beside me. I am also thankful to my other collaborators: Prof. Matteo Rizzi, Sebastian and Tapan for all the discussions regarding DMRG.

A special thanks to my friends of the Mathematics department, Vidyut and others for all the card games. I am also thankful to the family of Sambuddha Khan for giving me two wonderful months of fun and frolic last year. Finally, a special thanks to the Srivastav family; especially Ritu maam and Sanjay sir for all the special parties throughout the last three years.

A token of thanks to all the office staff of Department of Physics, IISc for their excellent cooperation. I am grateful to the Department of physics, IISc. for providing me with the Integrated Ph.D. scholarship and Center for Scientific and Industrial Research, India for generous financial support via JRF and SRF grants.
Publications based on this thesis

Original publications


Review articles


Synopsis

Ultracold atomic systems have provided an ideal platform to study the physics of strongly interacting many body systems in an unprecedentedly controlled and clean environment. And, since fermions are the building blocks of visible matter, being naturally motivated we focus on the physics of ultracold fermionic systems in this thesis. There have been many recent experimental developments in these systems such as the creation of synthetic gauge fields, realization of dimensional crossover and realization of systems with synthetic dimensions. These developments pose many open theoretical questions, some of which we address in this thesis.

We start the discussion by studying the spectral function of an ideal spin-$\frac{1}{2}$ Fermi gas in a harmonic trap in any dimensions. We discuss the performance of the local density approximation (LDA) in calculating the spectral function of the system by comparing it to exact numerical results. We show that the LDA gives better results for larger number of particles and in higher dimensions.

Fermionic systems with quasi two dimensional geometry are of great importance because of their connections to the high-{$T_c$} superconducting cuprate materials. Keeping this in mind, we consider a spin-$\frac{1}{2}$ fermionic system in three dimensions interacting with a contact interaction and confined by a one dimensional optical potential in one direction. Using the Bogoliubov-de Gennes formalism, we show that with increasing the depth of the optical potential the three dimensional superfluid evolves into a two dimensional one by looking at the shifts in the radio-frequency spectrum of the system and the change in the binding energy.
of the pairs that are formed.

The next topic of interest is studying the effect of synthetic gauge fields on the ultracold fermionic systems. We show that a synthetic non-Abelian Rashba type gauge field has experimentally observable signatures on the size and shape of a cloud of a system of non-interacting spin-$\frac{1}{2}$ Fermi system in a harmonic trap. Also, the synthetic gauge field in conjunction with the harmonic potential gives rise to ample possibilities of generating novel quantum Hamiltonians like the spherical geometry quantum Hall, magnetic monopoles etc.

We then address the physics of fermions in “synthetic dimensions”. The hyperfine states of atoms loaded in a one dimensional optical lattice can be used as an extra dimension, called the synthetic dimension (SD), by using Raman coupling. This way a finite strip Hofstadter model is realized with a tunable flux per plaquette. The experimental realization of the SD system is most naturally possible in systems which also have $\text{SU}(M)$ symmetric interactions between the fermions. The $\text{SU}(M)$ symmetric interactions manifest as long-ranged along the synthetic dimension and is the root cause of all the novel physics in these systems. This rich physics is revealed by a mapping of the Hamiltonian of the system to a system of particles interacting via an $\text{SU}(M)$ symmetric interaction under the influence of an $\text{SU}(M)$ Zeeman field and a non-Abelian $\text{SU}(M)$ gauge field. For example, this equivalence brings out the possibility of generating a non-local interaction between the particles at different sites; while the gauge filed mitigates the baryon ($\text{SU}(M)$ singlet $M$-body bound states) breaking effect of the Zeeman field. As a result, the site localized $\text{SU}(M)$ singlet baryon gets deformed and forms a “squished baryon”. Also, finite momentum dimers and resonance like states are formed in the system.

Many body physics in the SD system is then studied using both analytical and numerical (Density Matrix Renormalization Group) techniques. This study reveals fascinating possibilities such as the formation of Fulde-Ferrell-Larkin-Ovchinnikov states even without any “imbalance” and the possibility to evolve
a “ferromagnet” to a “superfluid” by the application of a magnetic field. Other novel fermionic phases with quasi-condensates of squished baryons are also demonstrated.

In summary, the topics addressed in this thesis demonstrate the possibilities and versatilities of the ultracold fermionic systems used in conjunction with synthetic gauge fields and dimensions.
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<th>Abbreviation</th>
<th>Full Form</th>
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<tbody>
<tr>
<td>BEC</td>
<td>Bose Einstein condensation/condensate</td>
</tr>
<tr>
<td>BCS</td>
<td>Bardeen Cooper Schrieffer</td>
</tr>
<tr>
<td>FFLO</td>
<td>Fulde-Ferrell-Larkin-Ovchinnikov</td>
</tr>
<tr>
<td>SF</td>
<td>superfluid</td>
</tr>
<tr>
<td>LDA</td>
<td>local density approximation</td>
</tr>
<tr>
<td>DOS</td>
<td>Density of states</td>
</tr>
<tr>
<td>SIDF</td>
<td>Spectral intensity distribution function</td>
</tr>
<tr>
<td>BDG</td>
<td>Bogoliubov-de Gennes</td>
</tr>
<tr>
<td>RF</td>
<td>radio frequency</td>
</tr>
<tr>
<td>BKT</td>
<td>Berezinskii-Kosterlitz-Thouless</td>
</tr>
<tr>
<td>GFC</td>
<td>gauge field configurations</td>
</tr>
<tr>
<td>S</td>
<td>spherical</td>
</tr>
<tr>
<td>EO</td>
<td>extreme oblate</td>
</tr>
<tr>
<td>EP</td>
<td>extreme prolate</td>
</tr>
<tr>
<td>PBC</td>
<td>periodic boundary condition</td>
</tr>
<tr>
<td>OBC</td>
<td>open boundary condition</td>
</tr>
<tr>
<td>ED</td>
<td>exact diagonalization</td>
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<tr>
<td>TBM</td>
<td>tight binding model</td>
</tr>
<tr>
<td>SD</td>
<td>Synthetic dimension</td>
</tr>
<tr>
<td>COM</td>
<td>Center of mass</td>
</tr>
<tr>
<td>PDOS</td>
<td>Pair density of states</td>
</tr>
<tr>
<td>GS</td>
<td>ground state</td>
</tr>
<tr>
<td>DMRG</td>
<td>density matrix renormalization group</td>
</tr>
<tr>
<td>PMDF</td>
<td>Pair momentum distribution function</td>
</tr>
<tr>
<td>CC</td>
<td>Calabrese-Cardy</td>
</tr>
<tr>
<td>EE</td>
<td>Entanglement entropy</td>
</tr>
<tr>
<td>SM</td>
<td>single component metal</td>
</tr>
<tr>
<td>DSF</td>
<td>dimer superfluid</td>
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Chapter 1

Introduction

1.1 Cold atom quantum simulators

Quantum matter is ubiquitous in nature and the constituent particles are quantum mechanical. Hence, the collective behavior of quantum mechanical particles is manifested in most of the important and intriguing problems of interest. Although the constituent particles may behave differently, collectively there arises many emergent spectacular phases because of the correlations amongst them. Many a times the exact models describing the physics of the correlated quantum systems are not known. Even when they are known, solving them exactly by analytical or numerical methods is often not possible. The reason behind this is the exponential increase in Hilbert space dimensions with the number of constituent particles. Hence, with the present computing power, the behavior of only very few number of particles can be studied. But since “More Is Different” [1], the accuracy achieved is often not sufficient to describe the physics under consideration.

Although a tremendous amount of experimental data for long standing problems like the high $T_c$ superconductivity exists, the physical understanding is far from complete. So, ideally physicists would want to model these data using theories and give predictions for further experiments. But since the accurate model for the system is not known, this program fails. Also, even when a simpler model is constructed capturing the essential physics of the system, exact solution of the model is not possible in most of the cases and microscopic quantities are not easy
1.2. ACHIEVEMENTS OF COLD ATOMS

to calculate. Hence, one can think in the opposite way. First, map a system Hamiltonian to an analog simulator and then the simulator can be prepared in a particular quantum state. This idea of quantum simulation using analog simulators which are otherwise inaccessible in naturally occurring systems goes back to Feynmann’s vision of quantum simulation [2]. The ultracold atomic systems provide this route.

Cold atoms provide the cleanest and unprecedentedly controlled environment for the simulation of natural systems with natural interactions. But, since these systems have very low temperatures (∼ 10 nK) and are artificial, it may seem that they have no connection to nature. Thanks to the astonishing developments in the field of atomic, molecular and quantum optics, new tools for studying spatio-temporal correlations of particles and characterization of physical data have emerged and cold atoms can now enter the quantum degeneracy regime both for bosonic and fermionic particles. Since in describing the properties of the ultracold atomic clouds, interaction plays a key role, the ideas of condensed matter physics come handy. Due to the huge success and extraordinary potential, the cold atomic systems have brought together a variety of seemingly diverging physical disciplines ranging from atomic, molecular and quantum optics, condensed matter physics, nuclear, gravitational, high energy physics, Quantum Chromodynamics (QCD) to quantum information theory.

1.2 Achievements of cold atoms

There were revolutionary developments in the field of atomic physics and quantum optics in terms of cooling and trapping of atoms, ions and molecules in the late 20th century and the 1997 Nobel prize in physics was awarded [3–5]. Evaporative cooling, together with these developments led to the experimental observation of the Bose-Einstein condensation (BEC) which was predicted theoretically long back by S. N. Bose and A. Einstein. This was also recognized by the Nobel committee in 2001 [6].

In a typical modern day cold atomic experiment, the typical temperature is ∼ 10 nK and the typical density of particles at the center of the atomic cloud is ∼ 10⁷–10⁹ m⁻³ [7]. These can be contrasted with the typical air density ∼ 10¹³ m⁻³
at room temperature \( \sim 300 \) K and atmospheric pressure and the typical density \( \sim 10^{16} \text{ m}^{-3} \) of liquids and solids. Hence, in addition to being ultracold, the cold atomic systems are ultra dilute. Due to the former quality, quantum effects start to manifest and the later gives rise to a simple effective description of the interaction between particles. Although these systems are extremely dilute, the interactions between the particles can be made large compared to other existing energy scales, leading to the possibility of simulating physics of strongly correlated systems [8].

The thrust in this direction of simulating strongly correlated systems in ultra-cold gases was provided by the theoretical proposal by Jaksh et al. [9] to study a superfluid (SF) to Mott insulator (MI) quantum phase transition by changing the depth of the optical lattice (see section 1.5) in which a BEC is loaded. Using this idea, the SF to MI phase transition was experimentally realized by Greiner et al. [10] even in the standard regime of average inter particle spacing much larger than the scattering length and this approach does not suffer from problems of condensate depletion due to three body losses. Several other experimental groups then observed this phenomenon in Bose-Fermi mixtures [11, 12]; even in molecules and bound repulsive pairs of atoms [13, 14].

Using the strong confinement produced by optical lattices several low dimensional systems have been achieved. The observations of a Tonks-Girardeau gas for hard-core bosons in one dimension [15–17] and a Berezinskii-Kosterlitz-Thouless phase transition [18, 19] constitute examples of achievements of strong correlations in low dimensional systems.

Solitary waves that propagate with constant velocity maintaining their shapes are called solitons. They are formed due to a compensation for the wave packet dispersion by nonlinear interaction and are found in a variety of systems including matter waves as solutions of the non-linear Schrödinger equation [20, 21]. Cold atomic systems, both bosonic [22, 23] and fermionic [24, 25], provide a controlled environment to study their properties, dynamics and collisions.

Phase transitions at zero temperature constitute a field of research where standard paradigms of condensed matter physics break down and has intrigued physicists over the past decade. Cold atomic systems can shed light into the physics of these quantum critical systems [26, 27]. Also, in spite of a huge effort devoted to the field of the disordered quantum systems over the past 50 years or so, there are
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many open questions and debates about the physics of interacting quantum systems with disorder. Realization of the Anderson model of disorder in cold atomic systems [28–31] has opened doors towards understanding and addressing these issues.

Recent developments in creating dipolar molecules [32] and Rydberg atoms [33–35] have opened up the possibility of tuning both nature and strength of the interaction between particles. A Rydberg atom is created by exciting a valence electron of an atom to a highly excited electronic state. As a result, a Rydberg atom has a radius of almost 1000 times that of the original atom and the interaction which governs the dynamics of these atoms is dipole-dipole interaction. Also, exciting new developments in realizing systems with $SU(N)$ symmetry [36–38] and achievement of quantum degeneracy regime for particles with large spins [39–41] has made it possible to study new classes of quantum many-body phenomena.

Experiments involving quantum quenches and engineered dissipative dynamics coupling the system to its environment, have enabled the study of far from equilibrium quantum dynamics [42–45]. Quantum transport experiments done using cold atomic systems have also connections to quantum transport in solids [46]. Non-equilibrium dynamics studied in table-top cold atomic experiments can shed light into the dynamics during very early stages of the universe [47].

Dynamical Abelian and non-Abelian gauge fields are of importance in the Standard model to understand the properties of nuclear matter and quark-gluon-plasmas. The quantum simulation of these gauge fields in cold atomic systems by clever use of the optical lattices is also possible [48–51]. Hopefully, these developments will lead to quantum simulation of QCD. Also, the ability to realize topological states is relevant to quantum information theory [52].

Overcoming some initial challenges like entropy removal (or the cooling problem), equilibration, probes and methods to characterize behavior, creation of gauge fields etc. [53], it is clear from the above discussion that these ultracold atomic systems have now emerged to be the cleanest controllable quantum emulators. Since, the interaction between the particles can also be tuned by changing magnetic field across a Feshbach resonance (see section 1.4), desired many body quantum Hamiltonians can be simulated and interesting quantum many body physics
both for continuum and lattice geometries in a controlled environment can be studied. This is otherwise difficult in natural systems, especially for strongly correlated systems of interest in many areas like condensed matter, nuclear physics, high-energy physics etc. The next section describes our motivation to concentrate on the physics of ultracold fermionic systems in this thesis.

### 1.3 Ultracold Fermi gases

The study of the ultracold Fermi gases is naturally motivated because the building blocks of visible matter are the fermions. As a result there exists a number of quantum many body interacting Fermi systems, e.g. superfluid $^3\text{He}$, superconductors, semiconductors, even astrophysical systems like the quark gluon plasma in the early universe, neutron stars etc. The study of the ultracold Fermi gases was also driven by the aim of creating a fermion condensate analogous to the BEC. The quantum degeneracy temperature ($T_d$) for both fermionic and bosonic systems can be estimated as $T_d \sim \frac{\hbar^2 n^{2/3}}{mk_B}$, with $n$ being the density of particles, $m$ is the mass of the atoms, $\hbar$ is the Planck’s constant and $k_B$ is the Boltzmann factor. But the emergence of this regime for fermionic system means a smooth crossover from a regime of classical to quantum behavior which is contrary to the bosonic case which is associated with the occurrence of a phase transition to the BEC state. For the fermionic case, superfluid phase can arise only for an interacting system which is not the case for bosons. In this dilute ultracold limit, the s-wave scattering in a single component Fermi gas is suppressed due to Pauli exclusion principle. This also suppresses three body losses in two component Fermi gas which is in contrary to the bosonic case [54].

An ultracold Fermi gas is created in the same way as that of an ultracold Bose gas [7]. That is, a beam of atomic vapor from an oven at temperature $\sim 500K$ is directed towards a Zeeman slower, in which the atomic beam is cooled to temperature $\sim 1K$ by using a laser beam propagating in the opposite direction. After the beam emerges from the Zeeman slower the atoms are captured by a magneto optical trap (MOT) and are further laser cooled to temperature $\sim 100\mu K$. In the MOT when the captured number of atoms are sufficient ($\sim 10^{10}$), a magnetic trap is turned on. Following this, the laser beams are turned off and the
magnetic trap only confines the atoms. An evaporative cooling is required to attain the quantum degeneracy temperature since at this stage the atomic density is very low. But due to the antisymmetry requirement in the total wavefunction, the evaporative cooling in this case is more challenging [55]. For constructing a model describing the system, how different parameters are related to experimental quantities can be illustrated by considering the following example. If we consider the magnetic trap to be a time averaged orbiting potential (TOP) trap which uses a superposition of a quadrupole magnetic field and a spatially uniform rotating magnetic field with the instantaneous magnetic field configuration

\[ B(t) = (B_1 x + B_0 \cos(\omega t), B_1 y + B_0 \sin(\omega t), -2B_1 z); \]

where, \( B_0 \) is the amplitude of the rotating field and \( B_1 \) is the amplitude of the quadrupole field. Then for an atom in a magnetic state labeled by \( j \), the magnetic contribution to energy can be written as

\[ E_j \sim E_j^0 - \mu_j B(x^2 + y^2 + 8z^2), \tag{1.1} \]

where, \( B = \frac{B_1^2}{4B_0}, \ E_j^0 = E_j(B_0) \) and \( \mu_j \) is the magnetic moment projected along the direction of the magnetic field. So, the magnetic field produces an anisotropic harmonic trapping potential with frequencies \( \omega_x^2 = \omega_y^2 = \frac{-2\mu_j B}{m} \) and \( \omega_z^2 = \frac{-16\mu_j B}{m} \). The anisotropy of the trap depends on the rotation axis of the bias field [7]. So, a natural starting point for discussing the physics of the ultracold Fermi systems is the ideal Fermi gas in the presence of a harmonic trapping potential (as considered in chapter 1).

The Feshbach resonance (section 1.4) can be used to tune the interaction between particles in a Fermi gas to several orders of magnitude because of the suppression of three body losses due to Pauli exclusion principle. The Bardeen-Cooper-Schrieffer (BCS) theory states that in the presence of a Fermi surface weakly attracting fermions can form cooper pairs and they condense to produce the BCS superconducting state. The cooper pairs have large spatial extent with their wave functions highly overlapping in space and hence, they can not be considered as composite bosons. Now, if we consider starting from this state and increasing the interaction between the fermions continuously, then for very large interaction strength two fermions with opposite spins will pair up and form tightly bound bosons. Then, these composite bosons which are made up of two fermions
can condense to produce a BEC state. This phenomenon (as depicted in fig. 1.1) of crossover from a BCS superfluid state of weakly bound Cooper pairs to the BEC state of strongly bound molecules of fermionic pairs is called the BCS-BEC crossover. This has been experimentally achieved in ultracold Fermi gases [56–58]. These studies have also been extended for imbalanced Fermi gases [59, 60] where there are different number of up and down species of fermions and there is a suppression of pairing due to Fermi surface mismatch.

Using techniques such as to understanding the dynamics of the quantum gas after its release from the trap, a direct measurement of the equation of state is possible and the full phase diagram can be mapped out [61–63]. By introducing lattice shaking which hybridizes different Bloch bands of the original optical lattice, ferromagnetic order can develop due to strong effective spin-spin interaction [64] or band structure with nontrivial topology [65] can be engineered. Recent developments in precision experiments to make a quantum gas microscope [66–68] have made it possible to push imaging resolution to the single site level. Also, a recently developed experimental technique called compensated optical lattice [69, 70] has enabled cooling of fermionic atoms very close to the Néel temperature.
1.4. Feshbach Resonance

In addition to being ultracold, the cold atomic systems are ultra dilute. The typical inter-particle distance is much larger than the effective range of the interaction. As a result, the dominant interactions present in these dilute and quantum degenerate gases are due to two body collisions. Also, two body collisions occur only when two particles come in the vicinity of one another and is typically $s$-wave in nature. This also ensures that the effective interaction potential has a simpler description which in many cases can be represented by a Dirac delta function. Hence, it is of contact type and its strength can be characterized by the $s$-wave scattering length. By using a phenomenon called the Feshbach resonance, described below, this scattering length can be tuned in cold atomic experiments by tuning a knob in the lab which controls the magnetic field.

Let us now consider a two-body collision. The set of quantum numbers describing the state of the two particles participating in the two-body collision is referred to as a channel. The entrance channel with which the particles come
close to each other is called the open channel, in the sense that it can have continuum states. If the other channel has total energy less than its threshold, it will be called a closed channel. In this way, this channel will not have sufficient energy to support continuum states of the two particles. A Feshbach resonance occurs when the total energy of two particles in an entrance channel, matches with the energy of a bound state in a closed or molecular channel (illustrated in fig. 1.2(a)).

The bound state energy in the closed channel can be tuned by tuning the magnetic field if the magnetic moment of the atom pairs in the two channels are different. Then it is possible to tune from a situation when the bound state is below the open channel threshold to a situation when the bound state is above the open channel threshold. Now, if there were no coupling between the two channels then the existence of bound state in the closed channel will not affect scattering in the open channel. But in practice, there will always be some small coupling (for example it can be induced by the exchange interaction) and as a result the scattering in the open channel is strongly modified. To first order in the coupling, the scattering in the open channel is not affected because of the absence of continuum states in the closed channel by definition. But to the second order, two particles in the open channel can go to a virtual intermediate state in the closed channel and come back. This process generates an effective interaction in the open channel which can be large when the total energy of the two particles in the open channel is close to the energy of the bound state in the closed channel. The s-wave scattering length $a_S$, thus generated by tuning magnetic field, has the following functional dependence on the magnetic field $B$

$$a_S(B) = a_{bg} \left(1 - \frac{\Delta}{B - B_0}\right),$$

where, $B_0$ is the value of the magnetic field when the resonance occurs and the scattering length diverges, $\Delta$ is called the width of the resonance and $a_{bg}$ is the scattering length away from the resonance [71].

An experimental realization of the Feshbach resonance in $^{40}$K Fermi gas is shown in fig. 1.2(b) (taken from ref. [55]). For Fermi gases, since the total wave function needs to be anti-symmetric, the s-wave collisions of same spin particles are suppressed. Another quantity which has important consequences is the width
1.5. OPTICAL LATTICES

of the resonance $\Delta$. When the width of the resonance is large (typically $\Delta \gg 1\text{G}$ [71]), the system can be described only in terms of the scattering length but for narrow resonances (typically $\Delta \ll 1\text{G}$ [71]) the effective range of the potential needs to be considered as well [72, 73]. Hence, to make a distinction between the two cases, $\Delta$ is not the only relevant parameter [71].

1.5 Optical lattices

When two or more laser fields are superimposed over each other so as to generate a standing wave, the corresponding electric field intensity becomes periodic in space. Since atoms couple with the laser field by an AC Stark effect, a periodic potential for the atoms can thus be created. The energy shift of an atom due to AC Stark shift in an electric field $E(r, t)$ of frequency $\omega$ is

$$\Delta \varepsilon = -\frac{1}{2} \alpha(\omega) \langle E(r, t)^2 \rangle ,$$

(1.3)

with $\alpha$ being the real part of the dynamical polarizability of the atom [7]. The time average over the square of the electric field is taken over a time which is much longer than the periodicity of the wave. The spatial periodicity of this part gives rise to the optical lattice.

For the generation of a $d$-dimensional hyper-cubic optical lattice, $d$ pairs of counter propagating laser beams from $d$ perpendicular directions, needs to be superimposed over each other. The total electric field intensity is then

$$E_{\text{tot}}(r, t) = \sum_{i=1}^{d} \left[ E_0^i \cos(q_i r_i - \omega t) + E_0^i \cos(-q_i r_i - \omega t) \right] ,$$

(1.4)

where the frequency of the lasers is $\omega$ and wavelengths $2\pi/q_i$. The total energy shift of the atom in this electric field (from eqn. (1.3)) is given by

$$\Delta \varepsilon = -2\alpha(\omega) \sum_{i=1}^{d} (E_0^i)^2 \cos^2(q_i r_i) .$$

(1.5)
Now, if electric field intensity amplitudes in all the $d$ directions are equal, i.e. $E_0^i = E_0; i = 1, \ldots, d$, then the optical potential has the form

$$V(r) = V_0 \sum_{i=1}^{d} \cos^2(q_i r_i),$$

(1.6)

where, $V_0 = -2\alpha(\omega)E_0^2$. This generates a static $d$-dimensional hypercubic optical lattice with period $a = \pi/q = \lambda/2$ with all the $q_i = q$ and $\lambda$ is the wavelength of the lasers.

Moving optical lattices can also be generated by superimposing beams with different frequencies [7]. By cleverly engineering the angle between different superimposed lasers or varying their number, it is possible to tune between optical lattices with different geometries as shown in fig. 1.3 (taken from the ref. [74]).

Figure 1.3: (Color online) Optical lattices with different geometries can be created by changing intensities of the laser beams. White and blue regions correspond to low and high optical potential regions respectively. This figure is taken from the ref. [74].
1.5. OPTICAL LATTICES

1.5.1 Energy bands of a 1d optical lattice

A relevant energy scale, which emerges due to the confinement of particles in a 1d optical lattice potential

$$V(z) = V_0 \cos^2 \left( \frac{\pi z}{a} \right) ,$$  
(1.7)

with $a$ being the lattice spacing, is the “recoil energy” ($E_R$) given by

$$E_R = \frac{\hbar^2 \pi^2}{2ma^2} ,$$  
(1.8)

where, $m$ is the mass of the particles. This is the ground state energy of a particle confined in a 1d box of length $a$. The single particle Schrödinger equation for a particle moving in the optical lattice (given by eqn. (1.7)) is

$$\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dz^2} + V(z) \right] \psi(z) = \epsilon \psi(z) ,$$  
(1.9)

with the single particle energies labeled by $\epsilon$. The above equation can be rewritten in the following way to have a standard form [75] of the Mathieu equation

$$\left[ \frac{d^2}{dz^2} + \{c - 2p \cos(2y)\} \right] \phi(y) = 0 ,$$  
(1.10)

where, $y = \frac{\pi z}{a}$, $c = \frac{ma^2(2\epsilon - V_0)}{\pi^2 \hbar^2}$, $p = \frac{ma^2 V_0}{2\pi^2 \hbar^2}$ and $\psi(z) = \psi\left( \frac{ay}{\pi} \right) = \phi(y)$. Hence, by standard solution of eqn. (1.10), the energy eigenvalues $\epsilon$ can be expressed in terms of Mathieu characteristic functions [75]. Applying the Bloch’s theorem to exploit the periodicity of the optical lattice, the energy bands $\epsilon_n(k)$ (labeled by the band index $n$ and expressed as a function of the crystal momentum $k$ in the reduced zone scheme) are given by

$$\epsilon_n(k) = \frac{G^2}{8} A \left( k + (-1)^{n-1} \left\{ n - \text{Mod}(n, 2) \frac{G}{2} \right\}, p \right) + \frac{V_0}{2} ,$$  
(1.11)

where, $G = \frac{2\pi}{a}$ is the reciprocal lattice vector and $A$ is the Mathieu Characteristic function.

In the next section, we discuss the importance and realizations of synthetic gauge fields in cold atomic systems.
1.6 Synthetic gauge fields

In modern quantum mechanics and quantum field theories, gauge theories play a central role in providing an understanding of the interactions between elementary particles. These gauge theories, which are associated with gauge fields, can also be emergent from another more complete theory. One of the simplest examples of the gauge theories is the theory of electromagnetism which is described by a scalar and a vector potential. The electric field arises due to temporal variation of the vector potential and spatial variation of the scalar potential, whereas the magnetic field is generated from the spatial variation of the vector potential only. Quantum matter in the presence of gauge fields show spectacular phenomenon and phases like the integer and fractional quantum Hall effects, topological insulators, vortices etc. Hence, it is absolutely essential to simulate the effect of electromagnetic fields in cold atomic systems.

But, since the atoms are neutral, they will not feel the effect of a direct electromagnetic field like the charged particles. Hence, to study the effect of gauge fields with cold atomic systems, their effects have to be simulated artificially such that the neutral atoms feel an effective gauge potential. This is why they are called

Figure 1.4: (Color online) (a) A typical experimental geometry for creating abelian artificial gauge fields with two counter propagating Raman laser beams linearly polarized in \( e_x - e_y \) plane with frequencies \( \omega \) and \( \omega + \delta \omega \) respectively. (b) Energy level diagram for atoms with total angular momentum \( F = 1 \), such as those relevant for alkali atoms like \(^7\)Li, \(^{23}\)Na, \(^{39}\)K, \(^{41}\)K and \(^{87}\)Rb. The quadratic Zeeman shifts are not taken into account. This figure is taken from the ref. [76].
synthetic gauge fields.

To create an effective synthetic gauge field, the conceptually simplest route is to use rotation. Since the Coriolis force experienced by a rotating atom has the same form as that of the Lorentz force experienced by a charged particle in a magnetic field, the frequency of rotation can be identified as the effective gauge field strength. This technique has been successfully used to generate vortices (which are hall-marks of the presence of magnetic field in a superfluid) in a superfluid BEC [77–79]. But since for a successful implementation of this technique, a quasi-exact balance condition between the Coriolis force and the trapping force must be satisfied, experimental implementation of this technique for systems with large number of particles is problematic. Also, due to technical limitations the simulated magnetic field is not very “large” to reach the quantum Hall regime.

Fortunately, there is another route provided by atom-light interaction exploiting the geometric Berry’s phase which appears as a result of adiabatically following one of the several internal states of a quantum particle [80–82]. This idea can be explained by considering an atom with two levels, a ground state \(|g\rangle\) and an excited state \(|e\rangle\), coupled by a laser beam. The eigenstates of this atom-light system are called dressed states which are linear combinations of \(|g\rangle\) and \(|e\rangle\). When the intensity and the phase of the light beam have spatial dependence, the dressed states also vary in space. Now, if the particle is initially prepared in one particular dressed state and adiabatically this state is followed, then there arises a Berry phase due to the spatial dependence of the dressed state. This Berry phase can be adjusted to mimic the Aharonov-Bohm phase experienced by a charged particle moving in a magnetic field. This idea has been successfully applied in experiments using a typical set up shown in fig. 1.4(a) with alkali atoms having energy levels shown in fig. 1.4(b) (fig. 1.4 is taken from the ref. [76]). Synthetic electric fields [83] and synthetic magnetic fields which led to the observation of vortices in ultracold bosonic systems and to the subsequent extension to generate spin-orbit coupling [84–87] have been achieved. Using these geometric gauge potentials, gauge fields in optical lattices can also be created [88–90] although by a different scheme than what is outlined here and the procedure is described in detail in the introduction of the chapter 5.

The synthetic gauge potential \((A)\), thus created is static and Abelian, i. e. dif-
ferent components of $A$ commute with each other. If different components of $A$
do not commute with each other and its gauge transformations can be described by elements of a non-commutative group, then $A$ is called a non-Abelian gauge field. A possible generation of these gauge fields can be done using the multipod scheme [91]. In this scheme, a single excited state is coupled by light to a set of quasi-degenerate ground states. This composite system provides a degenerate subspace and adiabatic motion in this subspace leads to the generation of non-Abelian gauge potentials. Recently, synthetic non-Abelian gauge fields have been realized in fermionic systems [92, 93].

Uniform static non-Abelian gauge fields have interesting physics for both bosonic [94–98] and fermionic systems [99–102] which will be taken up in chapter 4. Also, there has been many theoretical proposals for generating dynamical Abelian and non-Abelian gauge fields which are of importance in nuclear physics and QCD [48–51].

Next, we give an outline of the contents and a summary of the results of this thesis.

1.7 Outline of the thesis and summary of the results

With the motivations described in the previous sections for studying the physics of ultracold fermionic systems, we now give a detailed outline and summary of the results of this thesis in this section.

In chapter 2, we consider an ideal spin-$\frac{1}{2}$ Fermi gas confined in a harmonic potential, which is a natural starting point for discussing the physics of ultracold fermions. We study the performance of the local density approximation (LDA) in calculating dynamical response functions (here, we consider the spectral function) for this system by comparing with the exact results for different parameter regimes. The agreement between the exact and the LDA methods becomes better with the increase in dimensions and number of particles.

In the next chapter 3, we study the evolution of fermionic superfluid pairing as the dimension of the system is changed from three to two dimensions by tuning the depth of an optical potential applied in one direction. A Bogoliubov-de Gennes (BDG) theory, which accurately accounts for the periodic potential, is
constructed for the system of spin-$\frac{1}{2}$ fermions interacting in the singlet channel via a contact interaction and confined by an optical potential in the z-direction. The resulting BDG equations are then solved using Bloch’s theorem. For small potential depth, the linear responses of the density and the pairing gap of the system are obtained numerically and compared with the analytical perturbation theory calculations. The increase in the depth of the optical potential results in the division of the system into stacks of two dimensional layers with gradually decreasing inter-layer hopping and in the deep lattice limit the binding energy of the system is found to be in close agreement with that of the two dimensional result. The radio frequency spectrum of the system, which shows characteristic asymmetric dissociation peak structure and emergence of a clear pairing gap with increasing lattice depth (as seen in experiments), is calculated at any temperature.

The following chapters: chapter 4, chapter 5, chapter 6, chapter 7 and chapter 8 contain discussions of the physics of ultracold fermions in the presence of synthetic gauge fields both in continuum and lattice geometries.

On increasing the coupling strength ($\lambda$) of a non-Abelian gauge field that induces a generalized Rashba spin-orbit interaction, the topology of the Fermi surface of a homogeneous gas of non-interacting fermions undergoes a change. In chapter 4, we analyze how this phenomenon affects the size and shape of a cloud of spin-$\frac{1}{2}$ fermions trapped in a harmonic potential such as those used in cold atom experiments. We develop an adiabatic formulation, including the concomitant Pancharatnam-Berry phase effects, for the one particle states in the presence of a trapping potential and the gauge field, obtaining approximate analytical formulae for the energy levels for some high symmetry gauge field configurations of interest. An analysis based on the LDA reveals that, for a given number of particles, the cloud shrinks in a characteristic fashion with increasing $\lambda$. We explain the physical origins of this effect by a study of the stress tensor of the system. For an isotropic harmonic trap, the LDA predicts a spherical cloud even for anisotropic gauge field configurations. We show, via a calculation of the cloud shape using exact eigenstates, that for certain gauge field configurations there is a systematic and observable anisotropy in the cloud shape that increases with increasing gauge coupling $\lambda$. The reasons for this anisotropy are explained using the analytical energy levels obtained via the adiabatic approximation. These results should
be useful in the design of cold atom experiments with fermions in non-Abelian
gauge fields. An important spin-off of our adiabatic formulation is that it reveals
exciting possibilities for the cold-atom realization of interesting condensed mat-
ter Hamiltonians by using a non-Abelian gauge field in conjunction with another
potential. In particular, we show that the use of a spherical non-Abelian gauge
field with a harmonic trapping potential produces a monopole field giving rise
to a spherical geometry quantum Hall like Hamiltonian in the momentum repre-
sentation.

In chapter 5, first the idea of a “synthetic dimension” (SD) system, which en-
ables the realization of flux in an optical lattice in a “simpler” way, is introduced.
The synthetic dimension is generated using a standing wave optical potential on
a system of atoms having $M$ hyperfine states and inducing hopping along the hy-
perfine direction using Raman transitions. We study the single particle spectrum
of the system by exact diagonalization and it is compared with the dispersion
obtained using a tight binding model (TBM). We show that, for experimentally
relevant lattice depths, the bandwidths calculated using the two formulations
can differ by more than 20%. We, then, go on to discuss a TBM of the system in a
boosted frame and show that even the single particle SD system is very rich.

The most natural candidates for experimental realizations of the SD systems
also have $SU(M)$ symmetric interactions. Hence, in chapter 6, we consider the ef-
effect of an $SU(M)$ symmetric interaction in the two body sector of the SD system.
We discuss the two body physics of the system both for bosonic and fermionic
particles in a unified framework of $T$-matrix formalism. We show that due to the
interplay of the long-ranged $SU(M)$ symmetric interaction and the magnetic flux
present in the system novel phenomena such as the formation of finite momen-
tum dimers and resonance like states occur in the system.

In the following chapter 7, we investigate a generic few body SD system with
the $SU(M)$ symmetric interaction. A mapping to a system of multiflavor parti-
cles with an $SU(M)$ Zeeman field at each lattice site and a non-Abelian $SU(M)$
gauge potential that affects their hopping from one site to another transparently
demonstrates the few-body physics of the system. Using an exact diagonaliza-
tion technique in the k-space we analyze the problem. The site localized $M$-body
$SU(M)$ singlet bound state, called as “baryon”, is deformed to a nonlocal object
by the combination of the Zeeman field and the non-Abelian gauge potential. We also demonstrate that the system has a rich few-body phase diagram.

The novel phenomena occurring in the single particle and few body level of the interacting SD system clearly held promise of interesting many body physics in the system. In chapter 8, we investigate the many body physics of the SD system with $SU(M)$ symmetric interaction via analytical techniques like Bosonization and numerical finite system Density Matrix Renormalization Group (DMRG) method. We demonstrate that the system shows spectacular phenomena like the formation of FFLO states without “imbalance” and the possibility of generating a “superfluid” state starting from a “ferromagnetic” state. We also show the full many body phase diagram of the system containing a plethora of novel phases considering the $M = 2$ case as an example.

In the final chapter 9, we describe the exciting future directions of the results presented in this thesis.
Chapter 2

Spectral intensity distribution of trapped fermions

As discussed in the previous chapter, in recent years cold atomic systems have been a very active area of both theoretical and experimental research because of their ability to provide clean artificial systems and wide tunability of parameters. This huge success can widely be credited to the early developments in the field of atomic physics, in terms of laser cooling and trapping of atoms. Another major achievement of the last decade is development in the field of photo emission spectroscopy \[103, 104\]. Photo-emission spectroscopy enables us to probe occupied single-particle states of a fermionic system. The usual photo-emission spectroscopy was generalized to the case of cold atomic systems using radio frequency photo-emission spectroscopy \[105\]. In Fermi gases this technique reveals interesting many body effects like the pseudogap phenomenon \[106–109\]. In radio frequency photo-emission spectroscopy, the spectral function is routinely measured \[107\]. This spectral function is a very important quantity for a many body system since it reveals many important aspects like excitations of the system.

The static response properties of a many body system are well captured within LDA. But to account for the dynamics of the system in obtaining dynamical response functions, there are different versions of LDA like the time dependent LDA (TDLDA). Although, their applicability depends on the adiabaticity of the dynamics of the system \[110, 111\]. Let \(\chi(\omega^+, q)\) be a response function of a many
body system, where $q$ is momentum and $\omega$ is frequency, then to obtain the static response function, defined as $\lim_{\omega \to 0} \chi(\omega^+, q)$ or the average static response, defined as $\lim_{\omega \to 0} \lim_{q \to 0} \chi(\omega^+, q)$, LDA can be safely applied. But in obtaining the full dynamical response function $\chi(\omega^+, q)$, LDA should be applied with caution.

Even a system of non-interacting fermions in the presence of trap gives rise to many interesting behaviors since the trapping potential explicitly breaks spatial translational invariance [99, 112–115]. The chemical potential and thermodynamic quantities like the specific heat show oscillations as a function of number of particles due to the finite degeneracy of the energy eigenstates of the isotropic harmonic oscillator [116–118]. So, we take the viewpoint that this ideal Fermi system in the presence of an isotropic harmonic trapping potential is interesting and consider ideal spin-$\frac{1}{2}$ fermions at zero temperature with trap. We investigate the performance of LDA with respect to the exact case by calculating a dynamical response function of the system as an example. The dynamical response function, we calculate, is the spectral function of the many particle system, which we dubbed as spectral intensity distribution function (SIDF). This SIDF is calculated exactly and using LDA in any dimension for various parameter regimes. The results obtained using the two techniques are compared for different momenta, different trap frequencies and different number of particles in the physical dimensions 1, 2 and 3. We have found that as the dimension increases the agreement between the exact and LDA SIDFs gets better. Moreover, increase in number of particles makes LDA better in higher dimensions. In the following section, we describe the system under consideration and formulate the problem. The results of this chapter are from the ref. [119].

### 2.1 Description of the system

We consider a system of non-interacting spin-$\frac{1}{2}$ fermions trapped in a harmonic potential of frequency $\omega_0$. The Hamiltonian of the system is

\[
\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_0^2\hat{r}^2,
\]

(2.1)
2.2 Calculation of the SIDF

where \( m \) is the mass of fermions. Taking \( m \) and the Plank’s constant (\( \hbar \)) to be unity, the Hamiltonian operator in momentum space can be written as

\[
\hat{H} = -\frac{1}{2} \omega_0^2 \frac{\partial^2}{\partial \hat{p}^2} + \frac{1}{2} \hat{p}^2.
\] (2.2)

The one dimensional harmonic oscillator eigenstates in momentum space can be labeled by a quantum number \( n \) (\( n \) is an integer). They are

\[
\phi(n, k) = \frac{1}{\sqrt{2^n n!}} \left( \frac{1}{\pi \omega_0} \right)^{\frac{1}{4}} \exp \left( -\frac{k^2}{2\omega_0} \right) H_n \left( \frac{k}{\sqrt{\omega_0}} \right),
\] (2.3)

with energy \( \varepsilon_n = (n + \frac{1}{2})\omega_0 \). The eigenstates of the harmonic oscillator in \( d \)-dimension are products of \( d \) one dimensional harmonic oscillator eigenfunctions and are labeled by a set of \( d \) integers \( \{n_i : i = 1, 2 \ldots d\} \), denoted by the vector \( \mathbf{n} \).

In momentum space representation they are given by

\[
\langle k | \mathbf{n} \rangle = \psi(\mathbf{n}, k) = \prod_{i=1}^{d} \phi(n_i, k_i),
\] (2.4)

with energies \( \varepsilon(\mathbf{n}) = \sum_{i=1}^{d} (n_i + \frac{1}{2})\omega_0 \).

In calculating the SIDF exactly, these eigenfunctions are used while in LDA, the effect of the harmonic potential is taken approximately to obtain approximate local quantities.

2.2 Calculation of the SIDF

The single particle spectral function \( A(k, \omega) \) of a system measures the probability that a particle with certain momentum \( k \) has a particular energy \( \omega \). The SIDF \( I(k, \omega) \) for a many body system with finite density of particles is defined as the total contribution of all the particles to \( A(k, \omega) \). The SIDF is an important response function since it contains information about many body effects in the system. At zero temperature, the SIDF is calculated in this section using two techniques: A.) LDA and B.) Exact method. Complete analytical expressions for the SIDF is obtained using LDA in any dimension and exact SIDF is obtained numerically. We note that there are two important scales in the problem: trap center chemical po-
tential ($\mu_0$) and the frequency of the harmonic potential ($\omega_0$).

### 2.2.1 Calculation using LDA

Within LDA, we take into account the effect of the trap potential by making the chemical potential $\mu$ of the system a “local variable” [116]. This local chemical potential $\mu(r)$ determines the density of particles $\rho(r)$ at a point in space $r$ and they are related via the equation of state. As a function of $\mu_0$, $\mu(r)$ is given by

$$
\mu(r) + \frac{1}{2} \omega_0^2 r^2 = \mu_0 . \tag{2.5}
$$

The SIDF calculation within LDA is, therefore, two fold: first obtain $\mu_0$ as a function of $\omega_0$ and number of particles (N) and then calculate $\mu(r)$ to obtain SIDF.

#### Calculation of the trap center chemical potential

Consider there are $N$ particles in the trap which forms a cloud of radius $R_0$. Then in $d$-dimension ($d \geq 2$) we have

$$
\int_0^{R_0} \! dr \, \rho(r) = N , \tag{2.6}
$$

where $\rho(r)$ is the density of particles at position $r$. Considering spherical symmetry in the problem we note that $\rho(r)$ depends only on $r$, which leads to

$$
C(d) \int_0^{R_0} \! dr \, r^{d-1} \rho(r) = N , \tag{2.7}
$$

with $C(d) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ . The trap center chemical potential $\mu_0$ and the radius of the cloud $R_0$ is related as

$$
\mu_0 = \frac{1}{2} \omega_0^2 R_0^2 . \tag{2.8}
$$

If $g(k)$ is the DOS of non-interacting free spin-$\frac{1}{2}$ particles in $k$-space then $g(k) = 2 \frac{V}{(2\pi)^d}$ (the 2 factor comes from spin degeneracy and $V$ is volume of the $d$ spatial dimension) in $d$-dimension. So, for $d \geq 2$

$$
\int_0^{k_F} \! dk \, g(k) = N . \tag{2.9}
$$
2.2. CALCULATION OF THE SIDF

where \( k_F \) is the Fermi momentum of the system. The density of particles \( \rho \) is therefore

\[
\rho = \frac{2C(d)}{d(2\pi)^d} (2\mu)^{\frac{d}{2}},
\]

(2.10)

where \( \mu \) is the chemical potential of the free Fermi gas. This is the equation of state of the free Fermi gas in \( d(\geq 2) \) dimensions. Using this equation of state, from eqn. (2.5) and eqn. (2.10), within LDA the spatial density of the system can be written as

\[
\rho(r) = \frac{2C(d)}{d(2\pi)^d} (2\mu_0 - \omega_0^2 r^2)^{\frac{d}{2}}.
\]

(2.11)

Using eqn. (2.11) in eqn. (2.7) and then eqn. (2.8), we obtain a relation among the variables \( \mu_0, \omega_0 \) and \( N \) in \( d \)-dimension \((d \geq 2)\)

\[
\left( \frac{2\mu_0}{\omega_0} \right)^d = \frac{N}{X(d)},
\]

(2.12)

where \( X(d) = \frac{\sqrt{\pi}}{d(2^{d-1}) \Gamma \left( \frac{d}{2} \right) \Gamma \left( \frac{d+1}{2} \right)} \). Similarly, for a system of \( N \) particles in 1-dimension

\[
\int_{-k_F}^{k_F} dk \ g(k) = N,
\]

\[
\Rightarrow \sqrt{2\mu} = \frac{\pi \rho}{2}.
\]

(2.13)

The spatial density of the trapped particles \( \rho(r) \) obtained in the same way is

\[
\rho(r) = \frac{2\omega_0}{\pi} \sqrt{R_0^2 - r^2},
\]

(2.14)

and we have

\[
\int_{-R_0}^{R_0} dr \ \rho(r) = N.
\]

(2.15)

The above equation then leads to 1d version of eqn. (2.12) to be

\[
\frac{2\mu_0}{\omega_0} = N.
\]

(2.16)

Combining eqn. (2.12) and eqn. (2.16), the trap center chemical potential \( \mu_0 \), the trap frequency \( \omega_0 \) and the number of particles \( N \) in any dimension \( d \) are related as

\[
X(d)(\omega_0 R_0^2)^d = N.
\]

(2.17)
2.2. CALCULATION OF THE SIDF

This relation is used to calculate the local chemical potential.

Obtaining the SIDF within LDA

The single particle spectral function of a system with dispersion relation $\epsilon(k)$ is

$$A(k, \omega) = \delta(\omega - \epsilon(k)) \ .$$  \hspace{1cm} (2.18)

In LDA, the SIDF is obtained by adding local single particle spectral functions with suitable spectral weights. We approximate the free particle dispersion $\epsilon(k)$ to a local dispersion $\epsilon(r, k)$ at every position $r$ in space as

$$\epsilon(r, k) = \frac{k^2}{2} + \frac{1}{2} \omega_0^2 r^2 ,$$  \hspace{1cm} (2.19)

for the trapped system. The LDA form of the local single particle spectral function $A(r, k, \omega)$ is then

$$A(r, k, \omega) = \delta(\omega - \epsilon(r, k)) \ .$$  \hspace{1cm} (2.20)

So, the total SIDF $I(k, \omega)$ is obtained as

$$I(k, \omega) = \int dr \ W(r, k) A(r, k, \omega) ,$$  \hspace{1cm} (2.21)

where $W(r, k)$ is the spectral weight of the local single particle spectral function at position $r$. In $d$-dimension it is given by

$$W(r, k) = \frac{1}{(2\pi)^d} \Theta(\mu_0 - \epsilon(r, k)) \ .$$

In 1-dimensional, we use the relation between the trap center chemical potential $\mu_0$ and the number of particles $N$ from eqn. (2.16) in eqn. (2.21) to obtain

$$I(k, \omega) = \frac{1}{2\pi} \int_{-r_k}^{r_k} dr \ \delta(\omega - \frac{k^2}{2} - \frac{1}{2} \omega_0^2 r^2) \Theta(\mu_0 - \omega) ,$$  \hspace{1cm} (2.22)

where $r_k = \sqrt{\frac{2\mu_0 - k^2}{\omega_0}}$. This equation gives in 1-dimension the LDA-form of the SIDF

$$I(k, \omega) = \frac{\Theta(\mu_0 - \omega) \Theta(\omega - \frac{k^2}{2})}{\pi \omega_0 \sqrt{2\omega - k^2}} \ .$$  \hspace{1cm} (2.23)
In the same way as above, in $d$-dimension ($d \geq 2$) using eqn. (2.12) in eqn. (2.21) we get

$$I(k, \omega) = \frac{C(d)}{(2\pi)^d} \int_0^{r_k} dr \ r^{d-1} \delta(\omega - \frac{k^2}{2} - \frac{1}{2} \omega_0^2 r^2) \Theta(\mu_0 - \omega) \ , \quad (2.24)$$

which then gives the SIDF for spin-$\frac{1}{2}$ particles in $d$-dimension ($d \geq 2$) to be

$$I(k, \omega) = \frac{C(d)}{(2\pi)^d} \left( \frac{R_m}{\omega_0^2} \right)^{d-2} \Theta(\mu_0 - \omega) \Theta(\omega - \frac{k^2}{2}) \ , \quad (2.25)$$

where $R_m = \sqrt{\frac{2\omega-k^2}{\omega_0^2}}$.

The form of the SIDF $I(k, \omega)$ in any dimension $d$ from eqn. (2.23) and eqn. (2.25) can therefore be written as

$$I(k, \omega) = \frac{C(d)}{(2\pi)^d} \left( \frac{R_m}{\omega_0^2} \right)^{d-2} \Theta(\mu_0 - \omega) \Theta(\omega - \frac{k^2}{2}) \ , \quad (2.26)$$

where $R_m = \sqrt{\frac{2\omega-k^2}{\omega_0^2}}$ and $C(d) = \frac{2\pi^d}{\Gamma(\frac{d}{2})}$.

Due to the conservation of number of particles, the SIDF obeys the following sum rule

$$\int dk \ d\omega \ I(k, \omega) = N \ . \quad (2.27)$$

By using the expression of $I(k, \omega)$ from eqn. (2.26) and carrying out the integration over momentum and energy we see that the SIDFs obtained using LDA indeed obey the sum rule given by eqn. (2.27).

### 2.2.2 Exact calculation

To calculate the SIDF exactly we use the harmonic oscillator basis set given by eqn. (2.4). In terms of these eigenstates labeled by a set of quantum numbers $\{n\}$ with energies $\varepsilon(n)$, the SIDF is

$$I(k, \omega) = \sum_{\{n\}} |\langle n|k \rangle|^2 \delta(\omega - \varepsilon(n)) \ , \quad (2.28)$$

where the sum is over all the occupied states. The SIDF thus obtained is spherically symmetric. Orthonormality of the harmonic oscillator eigenfunctions can
be used to show that the sum rule given by eqn. (2.27) is satisfied. By using these eigenfunctions and carrying out the summation over all the occupied states for a fixed number of particles $N$, we obtain the SIDF in any dimension. The summation is performed numerically by broadening the delta function as a Lorentzian of ‘suitable width’. As an approximation to the delta function $\delta(x)$, we use the Cauchy-Lorentz probability density function

$$\mathcal{L}(x, \eta) = \frac{\eta}{\pi (x^2 + \eta^2)} ,$$

(2.29)

where $\eta$ is the half width at half maximum (HWHM) of $\mathcal{L}(x, \eta)$. As $\eta \to 0$, the Lorentzian distribution $\mathcal{L}(x, \eta)$ approaches the delta function distribution $\delta(x)$; where $\eta$ is a phenomenological parameter which needs to be chosen properly. Since the energy levels of an isotropic harmonic oscillator are degenerate, we account for suitable degeneracies of these levels and choose number of particles such that the highest occupied harmonic oscillator level is completely filled. The degeneracy $g^n_d$ of the $n$-th energy eigenstate of a $d$-dimensional isotropic harmonic oscillator can be calculated from the following recursion relation

$$g^n_d = \sum_{n_1=0}^{n} g^{n-n_1}_{d-1} \quad \text{with} \quad g^1_1 = 1 .$$

(2.30)

Using this relation, the expression for the degeneracy is obtained to be

$$g^n_d = \frac{\Gamma(n + d)}{\Gamma(d)\Gamma(n + 1)} ,$$

(2.31)

and if the highest occupied energy level is $n = \nu_{\text{max}}$, the total number of particles in the system (including spin degeneracy) is

$$N^n_{\nu_{\text{max}}} = 2 \sum_{n=0}^{\nu_{\text{max}}} g^n_d = \frac{2 \Gamma(\nu_{\text{max}} + d + 1)}{\Gamma(d + 1)\Gamma(\nu_{\text{max}} + 1)} .$$

(2.32)

The next section contains discussions of our results.
2.3 Results and discussion

It is noted from eqn. (2.26) that the definition of the SIDF at a particular momentum has behavior similar to the DOS of the system as a function of energy without the trapping potential at that particular dimension. So, we expect the SIDF of the trapped spin-$\frac{1}{2}$ fermionic system will have behavior similar to the corresponding DOS of the free spin-$\frac{1}{2}$ fermionic system. The DOS, $g(\omega)$, as a function of $\omega$ of a free spin-$\frac{1}{2}$ fermionic system in 1, 2 and 3 dimensions, has behaviors $\frac{1}{\sqrt{\omega}}$, constant and $\sqrt{\omega}$ respectively. Thus $I(k, \omega)$ of the trapped system at a particular momentum is expected to have similar behaviors in the corresponding dimensions. The dynamical response function, $I(k, \omega)$ encodes the dynamics of the system and have information about the excitations of the system. To capture the dynamical nature within LDA, there are different forms of LDA like the TDLDA but their domain of applicability is limited by the slowness of dynamics of the system [110, 111]. The simplest version of LDA (meaning the time independent LDA) is used here to calculate the dynamical response function $I(k, \omega)$ and its behavior is studied in different parameter regimes within this approximation. The SIDF obtained using LDA is compared with corresponding exact numerical results for different values of $k = |k|$ due to the spherical symmetry. We discuss their behaviors only for the physical dimensions 1, 2 and 3 although in principle the SIDFs can be calculated using these two methods in any dimension.

Owing to the limitation set by the trap center chemical potential and the conservation of energy, energy range of the LDA SIDF is limited to a certain window and it is non-zero only within this energy window. Although exact SIDF is large in this window but it has small spectral weight outside as well. Because of the energy time uncertainty principle, in a physical context with interaction if the excitations of the system have finite lifetime, then this leads to spectral broadening. The phenomenological broadening parameter can, therefore, be related to the finite lifetime of the quasi-particle excitations of the system. The minimum energy difference between the harmonic oscillator energy levels is $\omega_0$ and for comparing with the LDA SIDF, we have chosen the value of $\eta$ to be $\frac{\eta}{\omega_0} = \frac{1}{2}$ in exact calculation.

The comparison between the SIDFs in one dimension is shown in fig. 2.1 at
2.3. RESULTS AND DISCUSSION

different momenta \( k = 0 \) (fig. 2.1(a)) and \( k = \frac{k^0_F}{2} \) (fig. 2.1(b)) for \( N = 100, 250 \) and \( 500 \) which correspond to highest occupied energy level \( \nu_{max} = 49, 124 \) and \( 249 \) respectively. Here, \( k^0_F \) is the Fermi wave vector set by the trap center chemical potential \( \mu_0 = \frac{(k^0_F)^2}{2} \). We note that as the number of particles \( N \) in the trap

![Graph](image)

Figure 2.1: (Color online) The SIDF as a function of energy for a system of spin-\( \frac{1}{2} \) particles in one dimension with the broadening of the Lorentzian distribution \( \frac{\eta}{\omega_0} = 0.5 \) used in the exact calculation. The dotted curves denote exact results and the solid curves denote LDA results. (a) Shows the SIDF at zero momentum and (b) Shows the SIDF at a different value of momentum \( k = \frac{k^0_F}{2} \). The exact SIDF has spectral weights outside the LDA window and they have \( \frac{1}{\sqrt{\omega}} \) like dependence.
2.3. RESULTS AND DISCUSSION

Figure 2.2: (Color online) For a system of spin-$\frac{1}{2}$ fermions in two dimension, the behavior of the SIDF as a function of energy at a particular momentum with $\eta = 0.5$ used in exact calculation is shown. The dotted and the solid curves denote exact and LDA results respectively. In (a) SIDF for zero momentum and in (b) SIDF for finite momentum $k = k_F/2$ are shown. It is seen that the LDA SIDF is constant within a window but exact SIDF has weights outside that window as well. Smaller fluctuation in the exact SIDF is seen in finite momentum case.

is increased, the oscillations in the exact SIDF increases. With the change in momentum, the threshold for the SIDF in LDA result is expected to change because the free particle contribution to energy is changed but there is also a high frequency cutoff in the LDA spectrum due to the trap center chemical potential. The
2.3. RESULTS AND DISCUSSION

Figure 2.3: (Color online) The three dimensional SIDF as a function of energy at fixed momentum is shown. The value of the phenomenological parameter used in the exact calculation is $\eta_{\omega_0} = 0.5$ with exact and LDA results denoted by dotted and solid curves respectively. (a) Shows the SIDF at zero momentum and (b) Shows the SIDF at momentum $k = k_F^0/2$. We see that the SIDF has an overall $\sqrt{\omega}$ dependence and the exact spectra has weights outside the LDA window. The fluctuations in the exact SIDF is smaller for the case of finite momentum.

The exact SIDF has large weight within the LDA window and relatively small weight outside it; also at momentum $k = k_F^0/2$ but with suppressed fluctuations. We see from this figure that within corresponding window of energy the behaviors of the SIDFs for zero and finite momentum are similar.
2.3. RESULTS AND DISCUSSION

Figure 2.4: (Color online) Contour plot of the full SIDF spectrum as functions of both momentum and energy calculated using LDA is shown. (a), (b) and (c) show the spectra for 1, 2 and 3 dimensions for number of particles 500, 506 and 572 respectively. The SIDF corresponding to two dimension clearly shows that it is non-zero below a certain energy contour and is zero above it. The one dimensional SIDF for small momentum and large energies has smaller values but the three dimensional SIDF shows opposite behavior.

The expected constant behavior behavior of the SIDF in 2d is clearly seen from fig. 2.2. At zero and finite momentum the comparisons of exact and LDA SIDFs are shown in fig. 2.2(a) and fig. 2.2(b) respectively for number particles $N = 110$, $N = 506$ and $N = 1056$ which correspond to $\nu_{\text{max}} = 9$, 21 and 31 respectively. It is noted that at half of the Fermi momentum, LDA gives a better description than that of the zero momentum case. As expected the LDA window becomes smaller due to the increase in the free particle energy for the finite momentum case and the comparison with the exact results becomes better for larger number
of particles.

In 3d, $I(k, \omega)$ at a particular momentum $k$ as a function of $\omega$, has an overall $\sqrt{\omega}$ dependence which is as expected. This is seen form fig. 2.3 with number particles $N = 112$, $N = 572$ and $N = 1120$ which correspond to $\nu_{\text{max}} = 5$, 10 and 13 respectively at $k = 0$ (fig. 2.3(a)) and $k = k_0 F^2$ (fig. 2.3(b)). LDA SIDFs are in a particular window whereas exact spectra have some small spectral weights outside as well. The finite momentum SIDFs are similar to that of the zero momentum results although for finite momentum the oscillations get suppressed in the exact SIDF. In this dimension, as the number of particles in the system increases, LDA becomes better than that of 1d and 2d cases.

The oscillations seen in the exact SIDFs for a fixed number of particles can be attributed to the discrete nature of the harmonic oscillator spectrum and are closely related to Friedel oscillations. The periods of these oscillations vary with changing number of particles due to the change in the trap center chemical potential. Although for comparison we have chosen a particular value of $\eta = \frac{\omega_0}{2}$, with smaller values of $\eta$ the oscillations in the exact SIDFs increase and for very small values of $\eta$, the disagreement between the exact and LDA SIDFs increases and exact SIDFs become vanishingly small. We also see step like behavior when we calculate the exact SIDF as a function of $N$ at fixed $\omega$ and $|k|$, due to the finite degeneracy of the harmonic oscillator levels. The same kind of behaviors were also seen earlier in [117, 118] for this system in the thermodynamic quantities. They found that the thermodynamic quantities of this system oscillates as a function of $N$ and the continuum approximation or LDA gives better result at higher temperatures. But for very small temperature, the disagreement with the continuum approximation increases and the exact results become vanishingly small. So, the temperature there plays a similar role as the broadening parameter $\eta$ here.

In fig. 2.4, we show the full SIDF spectrum calculated using LDA in different dimensions 1, 2 and 3 (in the contour plots fig. 2.4(a), fig. 2.4(b) and fig. 2.4(c) respectively) as functions of both energy and momentum. The full spectrum has the same overall features as the fixed momentum case but now the behaviors at all momenta are seen. The two dimensional spectrum clearly shows a contour separating the zero and non-zero regions. The spectra corresponding to one and three dimensions have natures opposite to each other as is expected, meaning
the 1d spectrum has small spectral weights at small momenta and high energy whereas the 3d spectrum has large spectral weights in this regime.

By radio frequency photo emission spectroscopy technique, the spectral density function of a many body Fermi system is routinely measured \[107\]. Using spin polarized Fermi gas to load the atoms to a single hyperfine state, a non-interacting Fermi system can be created \[120\]. For RF measurements done in this kind of system of ultra dilute almost ideal Fermi gas, the SIDF of the system will have signatures as described earlier in this section. Hence, we see that although the exact dynamical response function \( I(k, \omega) \) has many interesting features, LDA captures essentially most of them quite accurately.

## 2.4 Summary

To calculate static response properties of a many body system, LDA can be safely applied. But its applicability is limited for the case of dynamical responses since dynamics of the system needs to be considered as well. To examine this, we have investigated how good is LDA in comparison to the exact case in determining the dynamical response functions of an ultracold many body Fermi system. We have considered a system of ultracold non-interacting spin-\( \frac{1}{2} \) Fermi gas confined by a harmonic trapping potential at zero temperature. Identifying important scales in the problem, we have calculated the SIDF of the system both exactly and approximately using the simplest version of LDA in any dimension and compared with each other for different dimensions, trap frequencies and momenta. The SIDFs at a particular momentum behave as the DOS of the system without trap in the corresponding dimension. We have shown that as the dimension increases the performance of LDA gets better and increase in number of particles makes LDA better in higher dimensions. Also LDA restricts the intensity distribution function to a certain range of energies but in contrary the exact spectra always has some small weights outside the LDA window fixed by momentum and the trap center chemical potential. We have also shown the full SIDF as a function of both energy and momentum calculated using LDA and expected behavior is seen.
Chapter 3

Theory of dimensional crossover of interacting fermions from three to two dimensions

Fermionic systems in quasi-2d geometry are of fundamental importance. This is mainly due to the following reasons. The origin of superconductivity in the high-$T_c$ superconducting materials (high-$T_c$ cuprates), which have the highest known superconducting transition temperature, has its origin in the physics of the quasi-2d cooper oxide planes. So, a deeper understanding of the physics of quasi-2d strongly interacting fermionic systems have potential to give physical insights into the problem of the high-$T_c$ superconductivity and its normal state. Also, since according to Mermin-Wagner theorem 2d is the marginal dimension for fermionic superfluidity, the study of quasi-2d systems are of great importance [121] to understand the role of the quasi-2d geometry in the fermion pairing within the superfluid.

In the introduction (section 1.3), we described that the phenomenon of BCS-BEC crossover is interesting in its own right. Hence, the study of BCS-BEC crossover in the quasi-2d geometry is naturally interesting. Recent developments in a series of cold atomic experiments [122–130] have made it possible to create these quasi-2d Fermi gases in a controlled manner and to measure their different interesting properties including the thermodynamic properties and the equation of state. These developments have bolstered theoretical research efforts [131–134] in
3.1. DESCRIPTION OF THE SYSTEM

We consider a system of spin-$\frac{1}{2}$ fermions interacting in the singlet channel via a contact interaction. The system is confined by an optical potential in the z-direction

$$V(z) = V_0 \sin^2\left(\frac{\pi z}{d}\right),$$

(3.1)

and the motion of the system in the other two directions is free. Here, $V_0$ is the depth of the optical lattice and $d$ is the separation between the minimums of two neighboring wells. Similar system but with a tight harmonic confinement in one direction instead of the optical potential has been considered earlier [132, 133] and our results match with those in the large $V_0$ limit.

The Hamiltonian of the system can be written as,

$$\mathcal{H} = \int d^3 r \left[ \frac{p^2}{2m} + V(z) - \mu \right] \psi_\sigma(r) + \int d^3 r \left[ \psi_\uparrow(r) \psi_\uparrow(r) \right],$$

(3.2)

where, $\sigma$ is the spin index ($\uparrow$ or $\downarrow$), $\psi_\uparrow(r) (\psi_\downarrow(r))$ is the fermion creation (annihilation) field operator which creates (annihilates) a fermion of flavor $\sigma$ at position $r$, $\mu$ is the chemical potential of the system determined by the number of particles and $g$ is the bare strength of the contact interaction. The first term in the above Hamiltonian is the kinetic energy part and the second term represents the two body contact interaction in the singlet channel.

A system of three dimensional superfluid in the presence of a periodic potential in one direction is expected to from coupled two dimensional layers with the inter-layer coupling dependent upon the depth of the potential. When the depth of the potential is very large so as to suppress the inter-layer tunneling, the system is expected to get divided into uncoupled layers. Thus, the system under consideration provides a unique opportunity to study the behavior of superfluidity in a layered environment with tunable interaction which also enables us to study
the BCS-BEC crossover. So, to investigate this dimensional crossover from 3d to 2d by tuning $V_0$, we formulate the problem by using the Bogoliubov-de Gennes formulation in the following section.

### 3.2 Bogoliubov-de Gennes formulation

Since the problem is not translational invariant due to the presence of the optical lattice potential, to study the problem of BCS to BEC crossover for this system within mean field theory (MFT), we resort to Bogoliubov-de Gennes (BDG) formulation. Within MFT, we rewrite the Hamiltonian (eqn. (3.2)) as

$$
\mathcal{H}_{BDG} = \int dr \left[ \psi_\sigma^\dagger(r) \left( \frac{p^2}{2} + V(r) - \mu \right) \psi_\sigma(r) + \Delta(r) \psi_\uparrow^\dagger(r) \psi_\downarrow(r) \right.
+ \left. \Delta^*(r) \psi_\downarrow(r) \psi_\uparrow(r) \right],
$$

where, $\Delta(r) = g \langle \psi_\downarrow(r) \psi_\uparrow(r) \rangle$ is the pair potential which breaks the quartic interaction term into two quadratic terms. Without loss of generality, it can be taken to be real since it is an expectation value over the ground state and the ground state is expected to have no currents. We work in the units $\hbar = m = 1$. This Hamiltonian $\mathcal{H}_{BDG}$ (eqn. (3.3)) is called the BDG Hamiltonian. $\mathcal{H}_{BDG}$ takes a more convenient form in the Nambu basis [135],

$$
\mathcal{H}_{BDG} = \int dr \begin{bmatrix} \psi_\uparrow(r) \\ \psi_\downarrow(r) \end{bmatrix} \begin{bmatrix} \mathcal{H}_0 & \Delta(r) \\ \Delta^*(r) & -\mathcal{H}_0 \end{bmatrix} \begin{bmatrix} \psi_\uparrow(r) \\ \psi_\downarrow(r) \end{bmatrix},
$$

with the Hamiltonian $\mathcal{H}_0 = \left( \frac{p^2}{2} + V(r) - \mu \right)$ is the non-interacting part and $\Psi(r) = \begin{bmatrix} \psi_\uparrow(r) \\ \psi_\downarrow(r) \end{bmatrix}$ is the Nambu operator. Now, we note that in this process, we are only left with a single particle Hamiltonian

$$
\mathcal{H}' = \begin{bmatrix} \mathcal{H}_0 & \Delta(r) \\ \Delta^*(r) & -\mathcal{H}_0 \end{bmatrix} = \mathcal{H}_0 \sigma_z + \Delta(r) \sigma_x,
$$

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3.2. BOGOLIUBOV-DE GENNES FORMULATION

which can be diagonalized and its eigen-system can be obtained. It is also noted that \( H_0 \) contains the periodic optical lattice potential. Since the potential is periodic in the \( z \)-direction (with a period \( d \)), we expect the gap function \( \Delta(r) \) also to be periodic in \( z \), i.e., \( \Delta(x, z + d) = \Delta(x, z) \) with \( \Delta(r) \equiv \Delta(x, z) \) and \( x \equiv (x, y) \). Then the full problem can be thought of as a single particle problem in a generalized periodic potential and the Schrödinger equation reads

\[
\left[ \left( \frac{p^2}{2} - \mu \right) \sigma_z + V(z)\sigma_z + \Delta(r)\sigma_x \right] \phi(r) = E\phi(r),
\]

where, the generalized periodic potential is the sum of the 2nd and 3rd terms in the left hand side of the above equation and the eigenfunctions \( \phi(r) \) are two component objects.

Exploiting the periodic nature of the underlying potential, Bloch’s theorem is applied to solve for the eigen-system of the problem. Since the system is translational invariant in the \( x - y \) plane but periodic in the \( z \)-direction, the eigen-system can be labeled by two indices \((n, k)\). The first index \( n \) labels the bands corresponding to a particular momentum \( k \). The momentum index \( k \equiv (\vec{k}_\perp, k_z) \) with \( k_z \) being the crystal momentum in the \( z \)-direction and \( \vec{k}_\perp \) being the in-plane \( (x - y \text{ plane}) \) momentum. The single particle Schrödinger equation, also known as the BDG equation, then takes the form

\[
\mathcal{H}' \phi_{n,k} = E_{n,k} \phi_{n,k},
\]

where the eigenfunctions are \( \phi_{n,k}(r) = \begin{bmatrix} u_{n,k}(r) \\ v_{n,k}(r) \end{bmatrix} \) with two components \( u_{n,k}(r) \) and \( v_{n,k}(r) \) with the energy eigenvalues \( E_{n,k} \). We also note that \( \mathcal{H}' \) has another set of eigen-system, given by

\[
\mathcal{H}' \tilde{\phi}_{n,k} = -E_{n,k} \tilde{\phi}_{n,k},
\]

where, \( \tilde{\phi}_{n,k}(r) = \begin{bmatrix} -v_{n,k}^*(r) \\ u_{n,k}^*(r) \end{bmatrix} \). Hence, the eigenvalues of \( \mathcal{H}' \) come in pairs \((E_{n,k}, -E_{n,k})\).

Now, we define the Bogoliubov quasi-particle creation operators \( \gamma_{n,k,+}^\dagger \) and \( \gamma_{n,k,-}^\dagger \) which create the quasi-particles in the states \( \phi_{n,k} \) and \( \tilde{\phi}_{n,k} \) respectively.
3.2. BOGOLIUBOV-DE GENNES FORMULATION

Then, the Nambu operator takes the form

$$\Psi(r) = \sum_{n,k} \begin{bmatrix} \phi_{n,k} & \tilde{\phi}_{n,k} \end{bmatrix} \begin{bmatrix} \gamma_{n,k,+} \\ \gamma_{n,k,-}^\dagger \end{bmatrix}$$

(3.9)

which gives,

$$\psi^\uparrow(r) = \sum_{n,k} \left( u_{n,k}(r) \gamma_{n,k,+} - v_{n,k}^*(r) \gamma_{n,k,-}^\dagger \right) ,$$

(3.10)

$$\psi^\downarrow(r) = \sum_{n,k} \left( u_{n,k}(r) \gamma_{n,k,-} + v_{n,k}^*(r) \gamma_{n,k,+}^\dagger \right) .$$

(3.11)

Hence, the above equation gives a mapping from the fermionic operators to the Bogoliubov quasi-particle operators. The pair potential then satisfies

$$\Delta(r) = g \sum_{n,k} u_{n,k}(r) v_{n,k}^*(r) \left[ 2n_F(E_{n,k}) - 1 \right] ,$$

(3.12)

where, $n_F(x) = 1/(e^{\beta x} + 1)$ is the Fermi function and $\beta$ is the inverse temperature in the units of the Boltzmann factor $k_B$. Similarly, the number density $\rho(r)$ of the system satisfies

$$\rho(r) = \sum_\sigma < \psi_\sigma(r) \psi_\sigma(r) >$$

$$\Rightarrow \rho(r) = 2 \sum_{n,k} \left[ |u_{n,k}(r)|^2 n_F(E_{n,k}) + |v_{n,k}(r)|^2 (1 - n_F(E_{n,k})) \right] .$$

(3.13)

So, to obtain the eigenspectrum of the system, the eqn. (3.7) has to be solved self-consistently together with eqn. (3.12) and eqn. (3.13).

But this problem described by the Hamiltonian $\mathcal{H}_{BDG}$ (in eqn. (3.3)) has ultraviolet divergence and renormalization is required by using an ultraviolet momentum cutoff $\Lambda$. To do so, the bare interaction parameter $g$ is traded with $\Lambda$ to obtain a physical parameter, the three dimensional $s$-wave scattering length $a_S$. The renormalization criterion can then be written as

$$\frac{1}{g} + \sum_{n,k} \frac{1}{2\epsilon_{n,k}^e} = \frac{1}{4\pi a_S} ,$$

(3.14)

where, $\epsilon_{n,k}^e$ are the empty lattice energy bands and $\Lambda = \sum_{n,k} \frac{1}{\epsilon_{n,k}^e}$ (see ref. [136]).
The Empty lattice approximation states that for a weak periodic potential, the approximate dispersion relation for a non-interacting particle moving in an “empty” crystal lattice is same as the energy dispersion of free particles. For translational invariance in the $x-y$ plane and lattice periodicity in the $z$-direction the empty lattice dispersion is given by

$$
\epsilon_{n,k}^e = \frac{k_\perp^2}{2} + \frac{1}{2} \left[ k_z + (-1)^{n-1} \{ n - \text{Mod}(n,2) \} G \frac{2}{d} \right]^2 ,
$$

(3.15)

with $G = \frac{2\pi}{d}$ being the reciprocal lattice vector of the optical lattice.

So, the BDG equation (eqn. (3.7)) has to be solved self-consistently along with the gap equation (eqn. (3.12)), the number equation (eqn. (3.13)) and the renormalization equation (eqn. (3.14)) to obtain the self-consistent solutions of the pair-potential and the chemical potential.

### 3.3 Solution of the BDG equation

The BDG equation (eqn. (3.7)) reads

$$
\begin{bmatrix}
\mathcal{H}_0 & \Delta(r) \\
\Delta(r) & -\mathcal{H}_0
\end{bmatrix}
\begin{bmatrix}
u_{n,k}(r) \\
u_{n,k}(r)
\end{bmatrix}
= E_{n,k}
\begin{bmatrix}
u_{n,k}(r) \\
u_{n,k}(r)
\end{bmatrix},
$$

(3.18)

Exploiting the in-plane translational invariance, we take an ansatz for the functions $u_{n,k}(r)$ and $v_{n,k}(r)$ to be

$$
u_{n,k}(r) = e^{i\vec{k}_\perp \cdot \vec{r}} u_{n,k_z}(z) ,$$

(3.16)

$$
v_{n,k}(r) = e^{i\vec{k}_\perp \cdot \vec{r}} v_{n,k_z}(z) ,$$

(3.17)

where, $A$ is the in-plane area of the system. Then, if $L_z$ be the spatial extent of the system in the $z$-direction, the volume of the system is $V = AL_z$. The BDG equation then takes the form

$$
\begin{bmatrix}
\mathcal{H}_0(\vec{k}_\perp) & \Delta(z) \\
\Delta(z) & -\mathcal{H}_0(\vec{k}_\perp)
\end{bmatrix}
\begin{bmatrix}
u_{n,k_z}(z) \\
u_{n,k_z}(z)
\end{bmatrix}
= E_{n,k}
\begin{bmatrix}
u_{n,k_z}(z) \\
u_{n,k_z}(z)
\end{bmatrix},
$$

(3.18)
3.3. SOLUTION OF THE BDG EQUATION

with \( \mathcal{H}_0^\perp(\vec{k}_\perp) = \frac{k^2}{2} - \frac{\partial^2}{\partial z^2} - \mu + V(z) \). We now define, \( \mathcal{H}_{\vec{k}_\perp}(z) = \begin{bmatrix} \mathcal{H}_0^\perp(\vec{k}_\perp) & \Delta(z) \\ \Delta(z) & -\mathcal{H}_0^\perp(\vec{k}_\perp) \end{bmatrix} \)

and \( \phi_{n,k_z}(z) = \begin{bmatrix} u_{n,k_z}(z) \\ v_{n,k_z}(z) \end{bmatrix} \). The Bloch’s theorem is then applied which gives

\[
\phi_{n,k_z}(z) = e^{ik_z z} \Phi_{n,k_z}(z),
\]

(3.19)

where, \( \Phi_{n,k_z}(z + d) = \Phi_{n,k_z}(z) \) are lattice periodic functions. Hence, in the Fourier representation we have

\[
\Phi_{n,k_z}(z) = \sum_m e^{imGz} \Phi_{n,k_z}(m),
\]

(3.20)

where, \( \Phi_{n,k_z}(m) \) are the Fourier components of the functions \( \Phi_{n,k_z}(z) \) labeled by the discrete index \( m \). The other periodic potentials, the optical lattice potential and the pair potential in Fourier space have the forms

\[
V(z) = \sum_m V_m e^{imGz},
\]

(3.21)

\[
\Delta(z) = \sum_m \Delta_m e^{imGz},
\]

(3.22)

with \( V_m \) and \( \Delta_m \) being the \( m \)-th Fourier components of the optical potential and the pair potential respectively. Also, since \( \Delta(z) \) is chosen to be real, from the above equation (eqn. (3.22)), it follows that \( \Delta_m = \Delta_{-m}, \forall m \).

Now, using eqn. (3.19) - eqn. (3.22) in the eigenvalue equation eqn. (3.18), we have

\[
\left[ \left( \frac{k^2}{2} - \mu \right) + \frac{(k_z + mG)^2}{2} \right] \sigma_z \Phi_{n,k_z}(m) + \sum_{m_1} (V_{m_1} \sigma_z + \Delta_{m_1} \sigma_x) \Phi_{n,k_z}(m - m_1) = E_{n,k} \Phi_{n,k_z}(m).
\]

(3.23)

Similarly, using the gap equation (eqn. (3.12)) and the number equation (eqn. (3.13)) respectively, we get the equations for the Fourier components of \( \Delta(z) \) to be

\[
\Delta(m) = g \sum_{m_1,n,k} u_{n,k}(m + m_1) v^*_{n,k}(m_1) \left[ 2n_F(E_{n,k}) - 1 \right],
\]

(3.24)
and of density $\rho(r)$ to be

$$
\rho(m) = \frac{2}{V} \sum_{m_1,n,k} v_{n,k}^*(m_1) u_{n,k}(m+m_1)n_F(E_{n,k}) + v_{n,k}^*(m_1)v_{n,k}(m+m_1)(1-n_F(E_{n,k}))
$$

(3.25)

If we are interested in the zero temperature properties of the system, the above two equations eqn. (3.24) and eqn. (3.25) can be further simplified by noting $n_F(E_{n,k}) = 0$ for this case.

### 3.4 Numerical scheme

The Hamiltonian (eqn. (3.7)) written in the Bloch basis, as discussed, becomes a single particle eigenvalue (eqn. (3.23)) problem. We solve this equation numerically, by choosing cylindrical geometry in the $k$-space. Here, $k_z$, the $z$-component of $k$ can take values within the first Brillouin zone and the corresponding reciprocal lattice vector is $G = \frac{2\pi}{a}$. For the in-plane component $\vec{k}$ of the momentum $k$, we choose its magnitude $|\vec{k}|$ to have a cutoff $k_c$ very large compared to the Fermi wave vector of the system. We also choose a cutoff $n_{cut}$ for the number of Bloch bands to be considered. We choose $n_{cut} = \frac{2k_c}{G}$ to be very large ($> 150$) and this in turn makes sure that the same regularization scheme (described in eqn. (3.14)) can be safely applied for all values of $V_0$ even in the *tight binding* limit. Then naturally, there is a restriction on the number of Fourier modes for the Bloch wave function to be used. If the Fourier modes of the wave function is labeled by $m$ and we use a cutoff $M_S$ such that $m$ takes values $m = -M_S, ..., 0, ..., M_S$ then the cutoff $n_{cut}$ used in the number of bands to be kept is $n_{cut} = (2M_S + 1)$. Also, if $m_1$ denotes the no. of Fourier modes for the pair potential, we use another cutoff $M$ such that $m_1 = -M, ..., 0, ..., M$ and we take $|M| < |M_S|$. In this truncated Bloch basis, the Hamiltonian is now numerically diagonalized and the eigenvalues $E_{n,k}$ with corresponding eigenfunctions $\phi_{n,k}$ are evaluated.

Self consistent solutions of the gap equation eqn. (3.24) and the number equation eqn. (3.25) are now needed to be obtained along with the renormalization condition given by eqn. (3.14). But the limits in the summations of these equations should be carefully chosen because of the cutoffs in the Fourier modes of
the wave function and in $|\vec{k}|$ denoted by $k_c$. The self consistent solutions of the Fourier modes of pair potential $\Delta(m)$ and the chemical potential $\mu$, thus obtained, can be used to calculate different physical quantities of interest like the binding energy, the radio frequency spectrum of the system etc. By varying the cutoffs, we make sure that the results do not depend upon the cutoffs, for sufficiently large cutoffs in the parameter regime of interest. To test our numerical scheme, we first obtain the linear response of the system to a small change in $V_0$, in the density and the pair potential analytically using perturbation theory and compare the perturbative results with the corresponding numerical results.

3.5 Perturbative calculation of linear response

When the confining potential $V(z)$ is weak, i.e. its depth $V_0$ is small (compared to the recoil energy $E_R = \frac{x^2}{2m}$), we expect a linear response of the system for a small change in $V_0$. To formulate a perturbation theory (closely following the ref. [135]), we define the unperturbed Hamiltonian as

$$H_0 = \int d^3r \, \psi^\dagger_\sigma(r) \left[ \frac{p^2}{2m} - \mu \right] \psi_\sigma(r) + g \int d^3r \, \psi^\dagger_\uparrow(r) \psi^\dagger_\downarrow(r) \psi_\downarrow(r) \psi_\uparrow(r), \quad (3.26)$$

and the perturbation term as

$$V = \int d^3r \, \psi^\dagger_\sigma(r) V(z) \psi_\sigma(r), \quad (3.27)$$

assuming $V_0$ to be a small parameter. The total Hamiltonian is $\mathcal{H} = H_0 + V$.

3.5.1 Linear response of a general operator

In this section, we formulate the calculation of the linear response of a general operator $A$ of the system described by eqn. (3.26) under the perturbation eqn. (3.27). Its response is defined as

$$\langle \Delta A \rangle = \langle A \rangle_{\mathcal{H}} - \langle A \rangle_0, \quad (3.28)$$
where, the expectation values are defined to be

\[ \langle A \rangle_H = \frac{\text{Tr} (e^{-\beta H} A)}{\text{Tr} (e^{-\beta H})}. \] (3.29)

\[ \langle A \rangle_0 = \frac{\text{Tr} (e^{-\beta H_0} A)}{\text{Tr} (e^{-\beta H_0})}. \] (3.30)

In the imaginary time formulation \( \tau = it \), the time dependence of the operator \( A \) is given by

\[ A(\tau) = e^{\tau H_0} A e^{-\tau H_0}. \] (3.31)

Defining, \( S(\tau) = e^{\tau H_0} e^{-\tau H} \), we have

\[ \frac{\partial S}{\partial \tau} = -V(\tau) S(\tau) \]

\[ \Rightarrow S(\beta) = T_\tau e^{-\int_0^\beta d\tau V(\tau)}. \] (3.32)

The expectation value of the operator \( A \), then becomes

\[ \langle A \rangle_H = \frac{\text{Tr} (e^{-\beta H_0} S(\beta) A)}{\text{Tr} (e^{-\beta H_0} S(\beta))}. \]

Assuming \( V(z) \) to be weak, we can now keep upto first order in \( V \) for the expansion in \( S(\beta) \) and rewrite the above equation as

\[ \langle A \rangle_H = \frac{\text{Tr} \left( e^{-\beta H_0} \left\{ 1 - \int_0^\beta d\tau V(\tau) \right\} A \right)}{\text{Tr} \left( e^{-\beta H_0} \left\{ 1 - \int_0^\beta d\tau V(\tau) \right\} \right)}, \]

\[ \Rightarrow \langle A \rangle_H = \frac{\left[ \text{Tr} (e^{-\beta H_0} A) - \int_0^\beta d\tau \text{Tr} (e^{-\beta H_0} AV(\tau)) \right]}{(1 - \beta \langle V \rangle_0) \text{Tr} (e^{-\beta H_0})}, \]

\[ \Rightarrow \langle A \rangle_H = \frac{\langle A \rangle_0 - \int_0^\beta d\tau \langle A(\tau)V \rangle_0}{1 - \beta \langle V \rangle_0}, \]

\[ \Rightarrow \langle \Delta A \rangle = \beta \langle A \rangle_0 \langle V \rangle_0 - \int_0^\beta d\tau \langle A(\tau)V \rangle_0. \] (3.33)

In the following sections, we use this result to calculate the linear responses of density and pair potential.
3.5. PERTURBATIVE CALCULATION OF LINEAR RESPONSE

3.5.2 Calculation of the density response

The confining potential is

\[ V(r) \equiv V(z) = V_0 \sin^2 \left( \frac{Gz}{2} \right). \]  

(3.34)

Thus, using eqn. (3.27) the perturbing potential operator is

\[ V = \sum_{q,k,\sigma} V_q C_{k,\sigma}^{\dagger} C_{k+q,\sigma}, \]

\[ \Rightarrow V = \sum_{k,q} \Psi_{k+q}^{\dagger} \Psi_{k+q} V_q, \]  

(3.35)

where,

\[ V_q = \int dr \, V(r) e^{i qr}, \]

\[ \Rightarrow V_q = \delta_{q_x,0} \delta_{q_y,0} \left[ \frac{V_0}{2} \delta_{q_z,0} - \frac{V_0}{4} (\delta_{q_z,G} + \delta_{q_z,-G}) \right] \]

(3.36)

and \( \Psi(k) \), the Nambu operator in the \( k \)-space, is

\[ \Psi(k) = \begin{bmatrix} C_{k,\uparrow} \\ C_{-k,\downarrow}^{\dagger} \end{bmatrix}. \]  

(3.37)

The density operator is

\[ \rho(r) = \sum_{\sigma} \psi_{\sigma}^{\dagger}(r) \psi_{\sigma}(r). \]  

(3.38)

The imaginary time dependent density operator can be written as

\[ \rho(r, \tau) = \sum_{q} e^{-i qr} \rho_q(\tau), \]

(3.39)

where,

\[ \rho_q(\tau) = \sum_{k,\sigma} C_{k,\sigma}^{\dagger}(\tau) C_{k+q,\sigma}(\tau), \]

\[ \Rightarrow \rho_q(\tau) = \sum_{k} \Psi_{k+q}^{\dagger}(\tau) \sigma_z \Psi_{k+q}(\tau). \]  

(3.40)
Then, using eqn. (3.33) the density response is formally written as
\[
\langle \Delta \rho(r) \rangle = \beta \langle V \rangle_0 \langle \rho(r) \rangle_0 - \int_0^\beta d\tau \langle \rho(r, \tau) V \rangle .
\] (3.41)

To calculate the right hand side of the above equation, we first calculate the Green’s function of the system. The Green’s function of the system is defined as
\[
\mathcal{G}(k, \tau) = - \langle T_\tau \Psi(k, \tau) \Psi^\dagger(k, \tau) \rangle ,
\] (3.42)
and its Fourier transform in terms of the fermionic Matsubara frequencies \[135\] is defined as
\[
\mathcal{G}(k, \tau) = \frac{1}{\beta} \sum_{i\nu_n} e^{-i\nu_n \tau} \mathcal{G}(k, i\nu_n) .
\] (3.43)

For the homogeneous system, the Green’s function is given by \[135\]
\[
\mathcal{G}_k(\omega^+) = \left[ \omega^+ \otimes 1 - \mathcal{H}_k \right]^{-1} ,
\] (3.44)
where, \[ \mathcal{H}_k = \begin{bmatrix} \xi_k & \Delta \\ \Delta & -\xi_k \end{bmatrix} , \] \[ \xi_k = \frac{k^2}{2m} - \mu \], \Delta is the gap and \( \mu \) is the chemical potential of the system. Then the Greens function becomes
\[
\mathcal{G}_k(\omega^+) = \frac{1}{\omega^{+2} - E_k^2} \begin{bmatrix} \omega^+ + \xi_k & \Delta \\ \Delta & \omega^+ - \xi_k \end{bmatrix} ,
\] (3.45)
where, \[ E_k = \sqrt{\xi_k^2 + \Delta^2} \] is the dispersion of the Bogoliubov quasi-particles. In terms of the fermionic Matsubara frequencies then
\[
\mathcal{G}_k(i\nu_n) = \frac{1}{i\nu_n^2 - E_k^2} \begin{bmatrix} i\nu_n + \xi_k & \Delta \\ \Delta & i\nu_n - \xi_k \end{bmatrix} .
\] (3.46)

To calculate the density response of the system within mean field theory, we now make use of this Green’s function. Using eqn. (3.35) and eqn. (3.39) in eqn. (3.41) we have
\[
\langle \Delta \rho(r) \rangle = \sum_{q, k} \int_0^\beta d\tau \ e^{-i q \cdot r} V_q \sum_{\alpha_1, \beta_1, \alpha_2, \beta_2} \sigma_{\alpha_1, \beta_1}^z \sigma_{\alpha_2, \beta_2}^z \mathcal{G}_{\beta_1, \alpha_2}(k + q, \tau) \mathcal{G}_{\alpha_1, \beta_2}(k, -\tau) ,
\]
3.5. PERTURBATIVE CALCULATION OF LINEAR RESPONSE

where, \( \alpha_i \) and \( \beta_i \) \((i = 1, 2)\) take values 1 and 2. The above equation can be compactly written as

\[
\langle \Delta \rho(r) \rangle = \sum_{q,k} e^{-iq \cdot r} V_q M(k, q),
\]

(3.47)

by defining

\[
M(k, q) = \frac{1}{\beta} \sum_{i \nu_n} \left[ \mathcal{G}_{11}(k, i \nu_n) \mathcal{G}_{11}(k + q, i \nu_n) - 2 \mathcal{G}_{12}(k; i \nu_n) \mathcal{G}_{12}(k + q, i \nu_n) \right. + \left. \mathcal{G}_{22}(k, i \nu_n) \mathcal{G}_{22}(k + q, i \nu_n) \right].
\]

(3.48)

Then using eqn. (3.46) we get

\[
M(k, q) = \frac{2}{\beta} \sum_{i \nu_n} \frac{1}{(i \nu_n)^2 - E_k^2} \left[ n_F(E_k + q) - n_F(-E_k) \right].
\]

(3.50)

To do the summations over the Matsubara frequencies, we have used the following identity

\[
\frac{2}{\beta} \sum_{i \nu_n} \frac{1}{(i \nu_n)^2 - E_k^2} = \left[ n_F(E_k) - n_F(-E_k) \right].
\]

(3.51)

which for \( T = 0 \) simplifies to

\[
M(k, q) = \frac{1}{\xi_{k + q} - \xi_k} \left[ \frac{\xi_k}{E_k} - \frac{\xi_{k + q}}{E_{k + q}} \right].
\]

(3.52)

We can now use these functions and calculate the Fourier modes of the density response using eqn. (3.47) and they are given by

\[
\langle \Delta \rho(m) \rangle = V_0 \sum_k \left[ \frac{1}{2} M(k, 0) \delta_{m,0} - \frac{1}{4} \left\{ M(k, (0, 0, G)) \delta_{m,1} + M(k, (0, 0, -G)) \delta_{m,-1} \right\} \right]
\]

(3.53)

Now, if we define

\[
\langle \Delta \rho(m = 1) \rangle = C_{\text{den}} V_0,
\]

(3.54)
3.5. PERTURBATIVE CALCULATION OF LINEAR RESPONSE

then we have

\[
C_{\text{den}} = \sum_k \frac{1}{4 (\xi_k - \xi_{k+q'})} \times \left[ \frac{\xi_{k+q'}}{E_{k+q'}} \left\{ n_F(E_{k+q'}) - n_F(-E_{k+q'}) \right\} - \frac{\xi_k}{E_k} \left\{ n_F(E_k) - n_F(-E_k) \right\} \right],
\]

(3.55)

where, \( q' \equiv (0, 0, G) \). The above expression at zero temperature simplifies to

\[
C_{\text{den}} = \sum_k \frac{1}{4 (\xi_k - \xi_{(k_z+G)})} \left[ \frac{\xi_k}{E_k} - \frac{\xi_{(k_z+G)}}{E_{(k_z+G)}} \right] .
\]

(3.56)

### 3.5.3 Calculation of the linear response to the pair potential

Similarly, here we calculate the linear response of the pair potential. The pair potential \( \Delta(r) \) is

\[
\Delta(r) = g \langle \psi_\downarrow(r) \psi_\uparrow(r) \rangle .
\]

(3.57)

With the imaginary time dependence, the pair potential in the Nambu representation can be written as

\[
\Delta(r, \tau) = g \sum_{k_1, k_2} e^{i(k_1 + k_2) \cdot r} \langle \Psi_{-k_1}^\dagger(\tau) \Theta \Psi_{k_2}(\tau) \rangle ,
\]

(3.58)

where, we have defined \( \Theta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \).

Using eqn. (3.33), the response to the pair potential is

\[
\langle d\Delta(r) \rangle = \beta \langle V \rangle_0 \langle \Delta(r) \rangle_0 - \int_0^\beta d\tau \langle \Delta(r, \tau) V \rangle .
\]

(3.59)

Along with eqn. (3.35) and eqn. (3.58) then the above eqn. (3.59) gives

\[
\langle d\Delta(r) \rangle = \sum_{q, k} \int_0^\beta d\tau \ e^{-iq \cdot r} V_q \sum_{\alpha_1, \beta_1, \alpha_2, \beta_2} \Theta_{\alpha_1, \beta_1} \sigma_{\alpha_2, \beta_2}^z G_{\alpha_1, \beta_2}(k + q, -\tau) G_{\beta_1, \alpha_2}(k, \tau) ,
\]

where, \( \alpha_i \) and \( \beta_i \) \((i = 1, 2)\) take values 1 and 2.

\[
\Rightarrow \langle d\Delta(r) \rangle = \sum_{q, k} e^{-iq \cdot r} V_q M_1(k, q) ,
\]

(3.60)
where,

\[ M_1(k, q) = \frac{1}{\beta} \sum_{i\nu_n} [G_{21}(k + q, i\nu_n)G_{11}(k, i\nu_n) - G_{22}(k + q, i\nu_n)G_{12}(k, i\nu_n)] , \]

\[ \Rightarrow M_1(k, q) = -\left( \frac{\Delta}{\beta} \right) \left( \frac{\xi_k + \xi_{k+q}}{E_{k+q}^2 - E_k^2} \right) \sum_{i\nu_n} \left\{ \frac{1}{(\nu_n)^2 - E_{k+q}^2} - \frac{1}{(\nu_n)^2 - E_k^2} \right\} , \]

\[ \Rightarrow M_1(k, q) = -\frac{\Delta}{2} \left( \frac{\xi_k + \xi_{k+q}}{E_{k+q}^2 - E_k^2} \right) \left[ \frac{n_F(E_{k+q}) - n_F(-E_{k+q})}{E_{k+q}} - \frac{n_F(E_k) - n_F(-E_k)}{E_k} \right] \]

which at \( T = 0 \) becomes

\[ M_1(k, q) = \frac{\Delta}{2E_k E_{k+q}} \left( \frac{\xi_k + \xi_{k+q}}{E_k + E_{k+q}} \right) . \] (3.61)

The Fourier modes of the pair potential are then given by

\[ \langle d\Delta(m) \rangle = V_0 \sum_k \left[ \frac{1}{2} M_1(k, 0)\delta_{m,0} - \frac{1}{4} \{ M_1(k, q')\delta_{m,1} + M_1(k, -q')\delta_{m,-1} \} \right] \] (3.62)

where, \( q' \equiv (0, 0, G) \). Defining,

\[ \langle d\Delta(m = \pm 1) \rangle = C_{del}V_0 \] (3.63)

we have

\[ C_{del} = \frac{g\Delta}{8} \sum_k \left( \frac{\xi_k + \xi_{k+q'}}{E_{k+q'}^2 - E_k^2} \right) \left[ \frac{n_F(E_{k+q'}) - n_F(-E_{k+q'})}{E_{k+q'}} - \frac{n_F(E_k) - n_F(-E_k)}{E_k} \right] \] (3.64)

which for \( T = 0 \) becomes

\[ C_{del} = \sum_k \left( \frac{g\Delta}{8E_k E_{k+q'}} \right) \left[ \frac{\xi_k + \xi_{k+q'}}{E_k + E_{k+q'}} \right] . \] (3.65)

### 3.5.4 Comparison of the linear responses

We now compare the linear responses of the system to small change in the potential depth at zero temperature obtained using perturbation theory described in the previous sections and the numerical scheme outlined in section 3.4. We show
3.6 Results and Discussion

In this section, we describe our numerical results. We calculate different properties of the system, specifically the binding energy of the “pairs” which are formed in the system and the radio-frequency spectrum of the system both in zero and finite temperature. To compare with the experiment [125], we choose similar parameters.

3.6.1 Calculation of the binding energy

The difference between the chemical potentials of the interacting system and that of the non-interacting system gives a measure of the binding energy of the “pairs” formed in the system. For a quasi two dimensional system having a “tight” har-
3.6. RESULTS AND DISCUSSION

Figure 3.2: Variation of the binding energy (defined as the difference between the chemical potentials of the interacting ($\mu$) and corresponding non-interacting system ($\mu_0$)) of the pairs as a function of $V_0$ for different values of the cutoff $k_c$ at resonance and at zero temperature.

Figure 3.3: Variation of the fitting parameters $a$ and $b$ with increasing cutoff $k_c$.

Harmonic confinement in one direction, the binding energy of the system is proportional to $\omega_0$ [137]. The frequency of the harmonic confinement is $\omega_0$ and for our system it can be estimated to be proportional to $\sqrt{V_0}$. We show the change in the binding energy as a function of the optical lattice depth in fig. 3.2 with $k_F d = 2.51$
at resonance ($\frac{1}{a_S} = 0$) and at zero temperature. We see that the binding energy decreases smoothly and it approaches $\sqrt{V_0}$ as $V_0$ increases. This reveals the two dimensional nature of the system for large $V_0$. This result is consistent with what was found in experiment [125]. To establish this more firmly, we fit the binding energy at large $V_0$ with a function $f(V_0) = a \times \left( \frac{V_0}{E_R} \right)^b$ where $a$ and $b$ are the fitting parameters. We show the variation of these fitting parameters as a function of the cutoff $k_c$ in fig. 3.3. We note that as expected the exponent $b$ approaches the value 0.5 as $k_c$ increases.

3.6.2 Radio frequency spectrum

In radio frequency (RF) spectroscopy, there is a momentum transfer from the photons to the atoms [121]. As a result, they get excited to empty hyperfine states. An analysis of the absorption intensity of the spectrum shows asymmetric line shapes corresponding to bound to free transitions of the atoms. This spectrum, which is called the RF spectrum, hence contains information about the binding energy of the atoms and by looking at the position of a peak in the RF spectrum, the binding energy of the pairs can be estimated.

![Figure 3.4: Behavior of the rf-threshold for different value of the width ($\eta$) of the Lorentzian at resonance and at zero temperature ($k_c = 20k_F$). Here, a.u. stands for arbitrary units.](image)
Formally, the RF transition operator $\hat{V}$ can be written as

$$\hat{V} = \sum_{n,k} \left( C_{n,k,3}^\dagger C_{n,k} + C_{n,k}^\dagger C_{n,k,3} \right), \quad (3.66)$$

where, $|n, k, 3\rangle$ denotes an initially empty excited hyperfine state [138]. An electron initially in the ground state in $|n, k\rangle$ gets excited to $|n, k, 3\rangle$. Then, the RF spectrum $\Gamma(\omega)$ [138] can be calculated using the Fermi’s golden rule and is given by

$$\Gamma(\omega) = \frac{2\pi}{\hbar} \sum_{n,k} |\langle n,k |\hat{V}| GS \rangle|^2 \delta(\hbar\omega - \hbar\Omega_{n,k}), \quad (3.67)$$

where, $\Omega_{n,k} = (E_{n,k} + \epsilon_{n,k} - \mu)$ and $\epsilon_{n,k}$ is the non-interacting energy dispersion. The spectrum thus becomes [138]

$$\Gamma(\omega) = \frac{2\pi}{\hbar} \sum_{n,k} |v_{n,k}|^2 \delta(\hbar\omega - \hbar\Omega_{n,k}). \quad (3.68)$$

To get a numerical estimate, we approximate the delta function as a Lorentzian of width $\eta$ and write it as

$$\delta(x) = \frac{\eta}{\pi(\eta^2 + x^2)}. \quad (3.69)$$

We now use eqn. (3.68) to numerically evaluate the rf-spectrum of the system.

Figure 3.5: (a) The radio frequency spectrum at resonance and at zero temperature with the momentum cutoff $k_c = 50k_F$ for different values of $V_0$. Here, a.u. stands for arbitrary units. (b) We define the value of $\omega$ at which the peak of the RF spectrum shown in (a) occurs as $\omega_{max}$. Then, we show the variation of the $\omega_{max}$ as a function of $V_0/E_R$. 

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3.6. RESULTS AND DISCUSSION

We define the rf-threshold ($\omega_{th}$) as the energy above which the rf-spectrum takes on non zero value. We know that the density of states of a superconductor diverges as its energy approaches the gap. From this behavior of the density of states, it is expected that the rf-spectrum should diverge at the rf-threshold for the uniform case ($V_0 = 0$). This behavior is seen in fig. 3.4 as $\eta$ is decreased. For the system at resonance and at zero temperature, we show the variation of the rf-spectrum of the system in fig. 3.5(a) for different values of $V_0$ and in fig. 3.5(b) we show the scaling of the peak position of the rf-spectrum as a function of $V_0$. We note that $\omega_{th}$ increases as $V_0$ is increased (consistent with the experiment [125]). This behavior is expected because $\omega_{th}$ gives a measure of the pair binding energy and it increases with increasing $V_0$. The RF-spectrum shows the characteristic asymmetric dissociation peak structure corresponding to the bound to free transitions of the atoms. In fig. 3.6, we now show the variation of the rf-spectrum with changing temperature at a large potential depth. It is noted that the rf-threshold decreases continuously with increasing temperature.

![Graph showing rf-spectrum at different temperatures](image)

**Figure 3.6:** The radio frequency spectrum at resonance and different temperatures at a large $V_0$ ($k_c = 30k_F$). Here, a.u. stands for arbitrary units.
3.7 Summary and conclusions

Motivated by recent experiments on the evolution of fermionic pairing across a crossover from three to two dimensions, we have analyzed the behavior of a system of spin-$\frac{1}{2}$ fermions with contact interaction in the singlet channel and confined in the z-direction by an optical potential while the motion in the other two directions is free. We have used a Bogoliubov-de Gennes mean field formulation that accurately accounts for the periodic potential inducing the dimensional crossover. As the depth of the potential is increased, the system gets transformed into a stack of two dimensional layers with tunneling between the layers gradually being suppressed. We solve the BdG equations numerically to obtain self-consistent solutions of the chemical potential and the density of the system by using a proper renormalization condition. They are used to further calculate different properties of the system such as the binding energy of the pairs and the radio frequency spectrum. To benchmark our code first we compare the linear responses of the system obtained numerically with those calculated using a perturbation theory. We show that as expected the system becomes two dimensional in the deep optical lattice limit by looking at the variation of the binding energy of the pairs formed in the system. The radio frequency spectrum of the system shows characteristic asymmetric dissociation peak structure and a clear pairing gap emerges with increasing lattice depth as seen in experiments. Similar results have been found in refs. [132, 133] but with an overall tight harmonic confinement which corresponds to the large $V_0$ limit.

An important future direction is to obtain the superfluid density tensor and study its variation with respect to the system parameters. This will be useful to understand the generalized Berezinskii-Kosterlitz-Thouless (BKT) transition occurring in the system as a function of the lattice depth.
Chapter 4

Trapped fermions in a synthetic non-Abelian gauge field

As discussed in the introduction section 1.6, synthetic gauge fields have interesting consequences on ultracold atomic systems. A spatially inhomogeneous Abelian gauge field produces non-trivial effects of a magnetic field, while a uniform Abelian gauge field is tantamount to a mere gauge transformation. On the other hand even a uniform non-Abelian gauge field produces interesting physical effects. In the context of bosons some aspects of these have been investigated and reported in the refs. [86, 96, 139].

A natural question, that arises, pertains to the effect of a uniform non-Abelian gauge field on fermions. Also, in fermions there are recent experimental realizations [92, 93] of non-Abelian gauge fields in 1D which bolster this research direction. A uniform non-Abelian gauge field induces a generalized Rashba spin-orbit interaction. The hint that such a system contains interesting physics came from ref. [100] which demonstrated that a highly symmetric (see below) non-Abelian gauge field induces a bound state between two fermions for any attractive interaction however small (small negative scattering length). This motivated the study [102] of the evolution of the many body fermionic ground state with the increasing strength $\lambda$ of the non-Abelian gauge field. The ground state realized for a weak attractive interaction in the absence of the gauge field is the BCS superfluid state with largely overlapping Cooper pairs. Ref. [102] demonstrated that on increasing the gauge coupling strength $\lambda$ the ground state evolves from
this just discussed BCS state to a BEC state, i.e., the non-Abelian gauge field induces a BCS-BEC crossover even at a fixed weak attraction that is unable to produce a bound state between two fermions in the absence of the gauge field. The most remarkable aspect of the crossover induced by the gauge field is that the BEC state obtained for large gauge couplings is made up of a new kind of tightly bound fermion-pairs called “rashbons” [102]. Hence, the condensate is called as rashbon-BEC (RBEC). Rashbons are anisotropic bosons characterized by an anisotropic dispersion and spin structure (nematicity), all of which are determined solely by the gauge field. The properties of RBEC, in particular the transition temperature, is determined by those of the corresponding rashbons. Properties of rashbons are reported in ref. [140], where it is demonstrated that a non-Abelian gauge field can enhance the exponentially small transition temperature of a Fermi gas with a weak attraction to the order of the Fermi temperature. Aspects of rashbon superfluidity, effects of Zeeman field and effects of imbalance etc. have been reported in refs. [101, 141-143].

A non-Abelian gauge field, also, has quite interesting effects on noninteracting fermions. In fact, for a fixed density $\rho$ of the particles, increasing the gauge coupling strength $\lambda$ induces a change in the topology of the Fermi surface at a critical gauge coupling strength $\lambda_T$. As shown in ref. [102], $\lambda_T$ is always of the order of $k_F$, the Fermi wave vector determined by the density ($\rho = k_F^3/(3\pi^2)$). In fact, in the presence of a weak attraction, the regime of gauge coupling over which the crossover from the BCS to RBEC takes place coincides with $\lambda \gtrsim \lambda_T$. In our opinion, one of the first things that cold atom experiments designed to study fermions in non-Abelian gauge fields should probe is this transition in the topology of the Fermi surface. The discussion above naturally gives rise to many interesting questions. Since almost all cold atom experiments are performed in a trap, it is useful to know the key signatures that provide measurable/falsifiable proof of the physics discussed above. A specific question is: How the presence of a generic non-Abelian gauge field influence the size and shape of a cloud of fermions? Answering this question is the key motivation of this chapter.

In this chapter, we investigate noninteracting fermions in a parabolic trapping potential characterized by a scale $\omega_0$, in presence of a uniform synthetic non-Abelian gauge field of strength $\lambda$. Our goal is to understand how a synthetic
4.1. PREAMBLE AND PROBLEM STATEMENT

non-Abelian gauge field affects the size and shape of a cloud of noninteracting fermions. To this end: (a) Noting that the most interesting regime corresponds to $\omega_0 \ll \lambda^2$, we develop a Born-Oppenheimer like approximation for the states of a trapped particle. This analysis reveals how the internal fast degree of freedom induces a Berry connection, i. e., a gauge potential on the motion of the particle, and most importantly suggests routes to generating interesting states such as the spherical geometry quantum Hall states using cold atoms. (b) We study the size and shape of the cloud of trapped noninteracting fermions under the influence of a non-Abelian gauge field using the LDA. (c) Finally, we compare the results of the LDA with exact numerical calculation. This last study reveals systematic and observable deviations from the LDA which are explained using the result of our adiabatic theory. This study finds that the cloud shrinks (consistent with ref. [101]). Importantly we uncover the scaling of the cloud size and the evolution of the anisotropy of the density profile with increasing gauge coupling, characteristic to various gauge field configurations of interest. We believe this will be of value for experiments on fermions in a non-Abelian gauge field. An important byproduct of our adiabatic analysis is the possibility of realization of interesting physics in cold atomic systems such as the spherical geometry quantum Hall state by using a synthetic non-Abelian gauge field in conjunction with another potential.

Section 4.1 consists of the background including notation, estimation of scales and sets up the statement of the problem. This is followed by a discussion of the one particle states in section 4.2 and subsection of 4.2.2 outlines the numerical method used to obtain exact one particle states for an extreme oblate (defined below) gauge field configuration. The effect of the gauge field on the cloud size and shape is discussed in section 4.3. The final section 4.4 consists of an itemized summary of the chapter. This chapter contains collaborative work done with Jayantha P. Vyasanakere and based on the ref. [99].

4.1 Preamble and problem statement

Denoting $\Psi^\dagger(r) = \{\Psi_\sigma(r)\}, \sigma = \uparrow, \downarrow$ as creation operators of spin-$\frac{1}{2}$ fermions at position $r$ in three spatial dimensions, the Hamiltonian under consideration [100,
is
\[ H_R = \int \mathrm{d}r \: \Psi^\dagger(r) \left( \frac{\hat{p}^2}{2} - \hat{p}_\lambda \cdot \tau \right) \Psi(r), \tag{4.1} \]
where \( \hat{p} \) is the momentum operator, \( 1 \) is the SU(2) identity, \( \tau^\mu \) (\( \mu = x, y, z \)) are Pauli matrices, \( \hat{p}_\lambda = \sum_i \hat{p}_i \lambda_i e_i \), \( e_i \)'s are the unit vectors in the \( i \)-th direction, \( i = x, y, z \). The gauge-field configuration (GFC) is described by a vector \( \lambda = \sum \lambda_i e_i \) and \( \lambda = |\lambda| \) is the gauge-coupling strength. We work in units where the Planck constant (\( \hbar \)) and the fermion mass \( m \) are set to unity. We are particularly interested in high symmetry GFCs called extreme oblate (EO) GFC with \( \lambda = \frac{\lambda}{\sqrt{2}} (e_x + e_y) \) and spherical (S) GFC with \( \lambda = \frac{\lambda}{\sqrt{3}} (e_x + e_y + e_z) \), which nurture interesting physics \[100, 102\].

The eigenstates of the noninteracting Hamiltonian (eqn. (4.1)) are
\[ |p\alpha\rangle = |p\rangle \otimes |\chi_\alpha(p)\rangle, \tag{4.2} \]
where \( |p\rangle \) is the plane wave state with momentum eigenvalue \( p \), \( \alpha = \pm 1 \) is the helicity with
\[ \hat{p}_\lambda \cdot \tau |\chi_\alpha(p)\rangle = \alpha |p\lambda\rangle |\chi_\alpha(p)\rangle. \tag{4.3} \]
In other words, \( |\chi_\alpha(p)\rangle \) is the eigenstate of \( \hat{p}_\lambda \cdot \tau \) in the spin sector with associated helicity \( \alpha \). The energy eigenvalues associated with the states in eqn. (4.2) are
\[ \varepsilon_\alpha(p) = \frac{p^2}{2} - \alpha |p\lambda|. \tag{4.4} \]

We now introduce an isotropic harmonic trapping potential with trapping frequency \( \omega_0 \) :
\[ \mathcal{H}_T = \frac{\omega_0^2}{2} \int \mathrm{d}r \: r^2 \Psi^\dagger(r) \Psi(r). \tag{4.5} \]
Unfortunately, the trapping potential spoils the symmetries of the system. Not only does it not commute with the usual kinetic energy term, it also does not let the helicity \( \alpha \) of eqn. (4.3) to be a good quantum number. This renders the diagonalization the full Hamiltonian
\[ \mathcal{H} = \mathcal{H}_R + \mathcal{H}_T \tag{4.6} \]
for a generic $\omega_0$ and $\lambda$ analytically intractable.

Fortunately, there are two well separated energy scales in the problem in the regime of interest. To see this, the typical traps used in experiments are such that the Fermi energy (set by the density of particles at the trap center) denoted as $E_F^0 = k_F^0/2$ and the trap frequency satisfies

$$\frac{E_F^0}{\omega_0} \approx 10^2. \quad (4.7)$$

We have also noted that the regime of interest of the gauge coupling corresponds to $\lambda \gtrsim \lambda_T$ where a change in the topology of the Fermi surface is engendered by the gauge field. In this regime of gauge coupling all the occupied states are of positive helicity. Noting that $\lambda_T \approx k_F^0$ for GFCs of interest, we see that the regime of interest corresponds to

$$\frac{\lambda^2}{\omega_0} \gtrsim 10^2. \quad (4.8)$$

In the next section, we show that the regime indicated in eqn. (4.8) allows us to make the adiabatic approximation and obtain the energy levels.

### 4.2 One Particle States – Adiabatic Theory

#### 4.2.1 Adiabatic Approximation for a Generic GFC

The development of an adiabatic approximation to diagonalize the Hamiltonian in eqn. (4.6) begins with the identification of the slow and fast degrees of freedom. This is most conveniently done in first quantized notation in momentum representation:

$$\mathcal{H} = -\frac{\omega_0^2}{2} \frac{\partial^2}{\partial p^2} + \frac{p^2}{2} - p_\lambda \cdot \tau, \quad (4.9)$$

where we have used $\hat{\mathbf{r}} = \hat{i} \frac{\partial}{\partial p}$, the position operator in momentum representation.

In the regime of interest (eqn. (4.8)), $\lambda \gg \sqrt{\omega_0}$, the internal spin degree of freedom is the “fast variable”, while the orbital degree of freedom is the “slow variable”. This is justified by the fact that the energy scale $\lambda$ that controls the dynamics of the internal spin degree of freedom is much larger than $\omega_0$ which controls the dy-
4.2. ONE PARTICLE STATES – ADIABATIC THEORY

Dynamics of the momentum variable\(^1\). This is analogous to the Born-Oppenheimer approximation in solid state and molecular systems where the electronic coordinates are the fast variables and ionic coordinates are the slow variables [144]; a detailed discussion of the adiabatic approximation can be found in ref. [145].

The above identification of the fast and slow degrees of freedom suggests an ansatz for the wave function of the particle

\[
|\psi\rangle = \int d\mathbf{p} \psi(\mathbf{p}) |\mathbf{p}\rangle \otimes |\chi_\alpha(\mathbf{p})\rangle.
\] (4.10)

The main physical content of this approximation is that the particle motion is such that with the change of \(\mathbf{p}\), it instantly attains internal spin state corresponding to a given helicity \(\alpha\) associated with \(\mathbf{p}\), hence its internal state is \(|\chi_\alpha(\mathbf{p})\rangle\). We shall restrict attention to \(\alpha = +1\) since this is the relevant case for \(\lambda \gtrsim \lambda_T\).

The goal now is to find the effective Hamiltonian for the wave function \(\psi(\mathbf{p})\). Since the spin Hilbert space has nontrivial topology, the associated Pancharatnam-Berry phase [146] induces a connection, which results in a new gauge field and a potential for the slow degree of freedom \(\mathbf{p}\). Following the line of analysis of ref. [146], we obtain the effective Hamiltonian as

\[
\mathcal{H}_{\text{eff}} = \frac{\omega_0^2}{2} \left( i \frac{\partial}{\partial \mathbf{p}} - A \right)^2 + \varepsilon(\mathbf{p}) + V_{BO}(\mathbf{p}),
\] (4.11)

where

\[
A = -i \langle \chi_\alpha(\mathbf{p}) | \frac{\partial \chi_\alpha(\mathbf{p})}{\partial \mathbf{p}} \rangle
\] (4.12)

is an induced connection (U(1) gauge potential) and

\[
V_{BO}(\mathbf{p}) = \frac{\omega_0^2}{2} \left( \langle \frac{\partial \chi_\alpha(\mathbf{p})}{\partial p_i} | \frac{\partial \chi_\alpha(\mathbf{p})}{\partial p_i} \rangle - \langle \frac{\partial \chi_\alpha(\mathbf{p})}{\partial p_i} | \chi_\alpha(\mathbf{p}) \rangle \langle \chi_\alpha(\mathbf{p}) | \frac{\partial \chi_\alpha(\mathbf{p})}{\partial p_i} \rangle \right)
\] (4.13)

is a potential where repeated spatial indices are summed. The energy levels are now obtained from the eigenvalue equation

\[
\mathcal{H}_{\text{eff}} \psi(\mathbf{p}) = \varepsilon \psi(\mathbf{p}).
\] (4.14)

\(^1\)For the spherical gauge field, for example, the Heisenberg equations of motion are \(\frac{\text{d} \mathbf{p}}{\text{d} t} = -\omega_0^2 \mathbf{\hat{r}}\) and \(\frac{\text{d} \tau}{\text{d} t} = -\frac{\lambda}{\sqrt{3}} \mathbf{\hat{p}} \times \tau\).
It is quite interesting to note that adiabatic motion in a non-Abelian gauge field results in a U(1) (Abelian) gauge field for the slow degree of freedom. Clearly, this promises to give rise to new possibilities with cold atom systems. We shall come back to this point in the sections that follow.

We now discuss application of the formulae developed in this section to various GFCs of interest.

### 4.2.2 Extreme oblate (EO) GFC

To solve for the levels of eqn. (4.14) for the EO GFC, we choose cylindrical polar coordinates in the momentum space \((p, \phi, p_z)\) with associated unit vectors \(e_p, e_\phi\) and \(e_z\). Here we consider the positive helicity state \((\alpha = +1)\):

\[
|\chi_+(p)\rangle \equiv \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ e^{i\phi} \end{array} \right).
\]  

(4.15)

We immediately obtain from eqn. (4.12) and eqn. (4.13) that

\[
A = \frac{1}{2p} e_\phi \quad \text{and} \quad V_{BO}(p) = \frac{\omega_0^2}{8p^2}.
\]  

(4.16)

The gauge field \(A\) corresponds to a half flux-quantum magnetic field line running along the \(z\) axis of the momentum space. The effective Hamiltonian from eqn. (4.11) reduces to

\[
\mathcal{H}_{\text{eff}} = -\frac{\omega_0^2}{2} \left( \frac{\partial^2}{\partial p^2} + \frac{1}{p} \frac{\partial}{\partial p} \right) + \frac{\omega_0^2}{2p^2} \left( L_z + \frac{1}{2} \right)^2 + \frac{p^2}{2} - \frac{\lambda}{\sqrt{2}} p + \frac{\omega_0^2}{8p^2} - \frac{\omega_0^2}{2} \frac{\partial^2}{\partial p_z^2} + \frac{p_z^2}{2},
\]  

(4.17)

where \(L_z = -i \frac{\partial}{\partial \phi}\) is the \(z\) component of the orbital angular momentum operator.

We use the separation of variable ansatz \(\psi(p, \phi, p_z) = \psi_p(p)\psi_\phi(\phi)\psi_z(z)\), and note that \(L_z\) eigenvalue can be labeled by an integer \(m\) to obtain

\[
-\frac{\omega_0^2}{2} u''(p) + \left[ \frac{\omega_0^2}{2p^2} \left( m + \frac{1}{2} \right)^2 + \frac{p^2}{2} - \frac{\lambda}{\sqrt{2}} p \right] u(p) = \tilde{\varepsilon} u(p),
\]  

(4.18)

where \(\psi_p(p) = \frac{u(p)}{\sqrt{p}}\) (a standard substitution), and \(\tilde{\varepsilon}\) is the “energy in the \((p, \phi)\) degrees of freedom”. This form is tailor-made for a semi-classical WKB treatment.
Figure 4.1: (Color online) One particle levels of a trapped particle in an EO GFC. Comparison of numerically calculated adiabatic WKB energies (dashed lines) obtained from the analysis of eqn. (4.18) and the exact result (solid lines in (a) and points in (b)) obtained using the method outlined in subsection of 4.2.2. Here $\Delta \varepsilon = \varepsilon(0, m, 0) + \frac{\lambda^2}{4} - \omega_0$. For $\lambda \gtrsim 5\sqrt{\omega_0}$ the results are indistinguishable with the approximate result of eqn. (4.20). (a) Energy for various values of $m$ quantum number as a function of $\lambda$. (b) Energy for $\lambda = 10\sqrt{2\omega_0}$ as a function of the quantum number $m$.

[147], which gives

$$\tilde{\varepsilon}(n, m) \approx -\frac{\lambda^2}{4} + \left(n + \frac{1}{2}\right) \omega_0 + \frac{\omega_0^2}{\lambda^2} \left(m + \frac{1}{2}\right)^2,$$  

(4.19)
where \( n \) is the “radial quantum number”. Remarkably, the presence of the potential induced by the connection obviates the need for the usual Langer modification \([147, 148]\) necessary in semi-classical analysis of this type and has the Maslov index of unity at both the turning points. Adding in the part of the energy from the \( z \) degree of freedom we obtain the energy eigenvalues of eqn. (4.17), dependent on three quantum numbers, to be

\[
\varepsilon(n, m, n_z) \approx -\frac{\lambda^2}{4} + (n + n_z + 1)\omega_0 + \frac{\omega_0^2}{\lambda^2} \left( m + \frac{1}{2} \right)^2. \tag{4.20}
\]

We now present a comparison of the result of eqn. (4.20) with the exact numerical solution of eqn. (4.6) briefly outlined in subsection of 4.2.2. Fig. 4.1(a) shows a plot of \( \Delta \varepsilon = \varepsilon(0, m, 0) + \frac{\lambda^2}{4} - \omega_0 \) obtained numerically (solid lines) and the same quantity calculated using full numerical WKB solution (dashed lines) of eqn. (4.18). This numerical WKB solution very closely matches the approximate analytical formula (eqn. (4.20)) for \( \lambda \gtrsim 5\sqrt{\omega_0} \). We see that the full numerical WKB solution has a quite remarkable agreement with the exact solution of eqn. (4.6) even for as small value of gauge coupling as \( \lambda = 4\sqrt{\omega_0} \) for \( m = 0 \), and agreement to even lower values of \( \lambda \) for higher values of \( m \).

Why does the adiabatic approximation work so well, and why does it work better for larger \( m \)? This can be understood by noting that the key ingredient of the adiabatic approximation is to “force” the particle to have a fixed helicity. This approximation is quite valid when the momentum of the particle is large (\( p \gtrsim \lambda \)). In the presence of the trapping potential, helicity fluctuations are maximum for momenta near the origin. Now note that the gauge potential induced by the Berry connection effectively provides for a nonzero centrifugal barrier for all \( m \) including \( m = 0 \) in momentum space (see eqn. (4.18), term proportional to \( (m + \frac{1}{2})^2 \)). This effectively “keeps the particle away from the origin” in momentum space, hence minimizing helicity fluctuations and rendering the adiabatic approximation accurate. The same argument provides the reason why the approximation works better for larger \( m \). In the discussion that follows we shall use eqn. (4.20).
Numerical Calculation of One Particle States for the EO GFC

In this subsection, we outline the method used to numerically obtain the exact one particle states for trapped fermions in an EO gauge field.

As is evident, we need to focus only on the in-plane degrees of freedom which are most conveniently described using polar coordinates \((r, \phi)\). The Hamiltonian (eqn. (4.6)) is rewritten in a more convenient form

\[
\mathcal{H} = \frac{\omega_0^2}{2} \frac{\hat{r}^2}{\mathcal{H}_0} + \frac{\lambda}{\sqrt{2}} \left( \hat{p}^+ \tau^- + \hat{p}^- \tau^+ \right),
\]

where \(\hat{p}^\pm = \hat{p}_x \pm i\hat{p}_y\) and \(\tau^\pm = \frac{1}{2}(\tau_x \pm i\tau_y)\). In the following, energy will be measured in the units of \(\omega_0\), distance in units of \(1/\sqrt{\omega_0}\), momentum and gauge coupling in the units of \(\sqrt{\omega_0}\), rendering all quantities dimensionless.

The eigenstates of \(\mathcal{H}_0\) are such that

\[
\mathcal{H}_0 |m, n, \sigma\rangle = (2n + |m| + 1)|m, n, \sigma\rangle,
\]

where \(m\) is the angular momentum quantum number, \(n\) is the radial quantum number and \(\sigma\) is the spin state which is \(\uparrow\) or \(\downarrow\), i.e.,

\[
|m, n, \sigma\rangle = |m, n\rangle \otimes |\sigma\rangle
\]

with

\[
\langle r, \phi | m, n \rangle = \sqrt{\frac{n!}{\pi(n + m)!}} e^{-\frac{r^2}{2}} r^m L_n^m(r^2)e^{im\phi},
\]

where \(L_n^m\) are associated Laguerre polynomials.

Since \(J_z = L_z + \frac{1}{2}\tau_z\) commutes with the Hamiltonian (eqn. (4.21)), we construct energy eigenstates which are also eigenstates of \(J_z\). Noting that the states \(|m, n, \uparrow\rangle\) and \(|m + 1, n, \downarrow\rangle\) are eigenstates of \(J_z\) with the eigenvalue \(m + \frac{1}{2}\), we can write the eigenstates of Hamiltonian (4.21) as

\[
|\Phi\rangle = \begin{pmatrix}
\sum_n a_{mn} |m, n, \uparrow\rangle \\
\sum_n b_{mn} |m + 1, n, \downarrow\rangle
\end{pmatrix},
\]
4.2. ONE PARTICLE STATES – ADIABATIC THEORY

where $a_{mn}$ and $b_{mn}$ are coefficients to be determined. This leads to a matrix eigenvalue problem which is completely defined with the specification of the following matrix elements:

\[
\begin{align*}
\langle m+1, n, \downarrow | \mathcal{H}_\lambda | m, n' \uparrow \rangle &= -\frac{i \lambda}{\sqrt{2}} \left( \sqrt{m+n+1} \delta_{n,n'} + \sqrt{n+1} \delta_{n,n'-1} \right), \\
\langle m, n, \uparrow | \mathcal{H}_\lambda | m+1, n' \downarrow \rangle &= i \frac{\lambda}{\sqrt{2}} \left( \sqrt{m+n'+1} \delta_{n,n'} + \sqrt{n'+1} \delta_{n',n-1} \right). \tag{4.26}
\end{align*}
\]

In actual numerical calculations, the sum over $n$ in eqn. (4.25) has to be truncated at some large value $N$. We have chosen $N$ in such a way that the energy eigenvalues are converged to the desired double precision accuracy. When $\lambda = 20\sqrt{2}$, we used $N = 180$ to obtain this accuracy.

4.2.3 Spherical (S) GFC

With the reassuring success of the adiabatic approximation for the EO GFC, we now turn to the spherical GFC. The natural coordinate system to treat this GFC via eqn. (4.11) is the spherical polar coordinate system in momentum space with $(p, \theta, \phi)$ as the coordinates and $e_p, e_\theta$ and $e_\phi$ as the basis vectors. The positive helicity eigenstates in this basis are

\[
|\chi_+ (p) \rangle \equiv \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}.
\tag{4.27}
\]

It follows from eqn. (4.12) and eqn. (4.13) that

\[
A = -\frac{\cot \theta}{2p} e_\phi \quad \text{and} \quad V_{BO}(p) = \frac{\omega_0^2}{4p^2}. \tag{4.28}
\]

Quite interestingly, the gauge potential $A$ corresponds to the presence of a monopole of charge $Q = \frac{1}{2}$ at the origin of the momentum space. The angular motion (i.e., $(\theta, \phi)$) of a particle in a monopole field has been extensively studied \[149, 150\] and also used in the context of quantum Hall effect in the so called spherical geometry (see
section 3.10 of \([151]\)). We thus obtain,

\[
\mathcal{H}_{\text{eff}} = -\frac{\omega_0^2}{2} \left( \frac{1}{p^2} \frac{\partial}{\partial p} p^2 \frac{\partial}{\partial p} \right) + \frac{\omega_0^2}{2p^2} \left[ -\frac{1}{\sin \theta \frac{\partial}{\partial \theta}} \sin \theta \frac{\partial}{\partial \theta} + \left( Q \cot \theta + \frac{i}{\sin \theta \frac{\partial}{\partial \phi}} \right)^2 \right]
+ \frac{\omega_0^2}{4p^2} + \left( \frac{p^2}{2} - \frac{\lambda}{\sqrt{3}} \right).
\]

\[(4.29)\]

Again using a separation of variable ansatz, \(\psi(p, \theta, \phi) = \psi_p(p) \Omega(\theta, \phi)\), and noting that the eigenvalues of the angular operator in the square bracket in eqn. \((4.29)\) is \(\ell(\ell + 1) - Q^2\) with \(\ell = |Q|, |Q| + 1, \ldots\) where \(\ell\) is the angular momentum quantum number. For each \(\ell\) there are \(2\ell + 1\) states with magnetic quantum numbers \(m = -\ell, -\ell + 1, \ldots, \ell\), and the wave function \(\Omega(\theta, \phi)\) is one of the monopole har-
monics [151]. We thus obtain

\[-\frac{\omega_0^2}{2} \left( \frac{1}{p^2} \frac{\partial}{\partial p} p^2 \frac{\partial \psi_p}{\partial p} \right) + \left[ \frac{\omega_0^2}{2p^2} \left( (\ell + 1) - |Q|^2 + \frac{1}{2} \right) + \left( \frac{p^2}{2} - \frac{\lambda}{\sqrt{3}} p \right) \right] \psi_p = \varepsilon \psi_p ,\]

which is amenable to a semi-classical WKB treatment with \( \psi_p(p) = u(p)/p \). An approximate analysis gives us

\[\varepsilon(n, \ell, m) \approx -\frac{\lambda^2}{6} + (n + \frac{1}{2}) \omega_0 + \frac{3\omega_0^2}{2\lambda^2} \left( \ell(\ell + 1) + \frac{1}{4} \right) .\]

This result, again, is obtained without the usual Langer modification applied to WKB treatment of three dimensional problems [148].

A key inference that can be made from the results of this section is that non-Abelian gauge fields used in conjunction with a trapping potential can give rise to many new possibilities with cold atoms. In particular when the gauge coupling is strong compared to the trapping potential, the motion of the particle is adiabatic with respect to the spin degree of freedom as discussed in detail at the beginning of subsection 4.2.1. The associated Pancharatnam-Berry phase produces an effective gauge field which can be used to realize Hamiltonians that are quite interesting. In this regard, it is very interesting to note that the spherical non-Abelian gauge field in conjunction with a harmonic trapping potential (just discussed) produces a magnetic monopole (shown in fig. 4.2) and “spherical geometry” quantum Hall like Hamiltonian in the momentum space.

The discussion above, quite clearly, brings out a variety of possibilities of generating interesting Hamiltonians when a non-Abelian gauge field is used in conjunction with a spatially varying potential \( V(r) \). Furthermore, this discussion is entirely based on one body physics and will equally well apply to bosons.

### 4.3 Sizes and Shapes of Trapped Clouds

In this section, we investigate the natural question that arises: Is there any discernible effect on the size and shape of the trapped clouds due to the influence of the non-Abelian gauge field? In the following subsection, we answer this question using the LDA and compare it with the exact calculation in the subsection.
4.3. SIZES AND SHAPES OF TRAPPED CLOUDS

Figure 4.3: (Color online) Dependence of the radius of the cloud on the gauge coupling $\lambda$ for the EO GFC. $R_0$ is the radius of the cloud in the absence of the gauge field and $k_F^0$ is the Fermi wave vector related to the trap center density in the absence of the gauge field.

that follows. We restrict our attention to zero temperature.

4.3.1 Local Density Approximation

Similar to the eqn. (2.5) in chapter 2, within the LDA [7], the spatial dependence of the chemical potential $\mu$ is determined by the trapping potential,

$$\mu(r) + \frac{1}{2} \omega_0^2 r^2 = \mu_0 ,$$

(4.32)

where $\mu_0$ is the chemical potential at the trap center, and $\mu(r)$ is the “local chemical potential” that determines the density $\rho(r)$ of fermions at the point $r$. It follows from eqn. (4.32) that within LDA, $\mu(r)$ and hence $\rho(r)$ will be dependent only on $r = |r|$. The density $\rho$ is related to the chemical potential $\mu$ by the equation of state. For the GFCs of interest, we can obtain the equation of state analytically [152]. We shall focus mainly on the EO GFC for which the equation of state
is given by

\[
\rho(\mu) = \begin{cases} 
\frac{(2\mu)^{3/2}}{\sqrt{2\pi}} + \frac{\lambda}{2\sqrt{2\pi}} \left( \sqrt{\mu + \left( \frac{\lambda^2}{2} + 2\mu \right) \sin^{-1} \left( \frac{\lambda}{\sqrt{4\mu + \lambda^2}} \right)} \right) & \text{if } \mu \geq 0, \\
\frac{\lambda(4\mu + \lambda^2)}{8\sqrt{2\pi}} & \text{if } -\frac{\lambda^2}{4} \leq \mu < 0.
\end{cases}
\]

The system is such that the radius of the cloud at \( \lambda = 0 \) is \( R_0 \), with a trap center density given by \( \rho_0 = \left( \frac{k_0 F}{\omega_0} \right)^3/3\pi^2 \) and an associated Fermi energy \( E_F^0 = \left( \frac{k_F^0}{2} \right)^2/2 = \frac{1}{2}\omega_0 R_0^2 \). On application of the gauge field, the radius of the cloud changes to \( R \). We seek to determine the dependence of \( R \) on \( \lambda \). This is determined from the equation

\[
4\pi \int_0^R dr \, r^2 \rho(\mu(r)) = \frac{1}{24} R_0^6 \omega_0^3, \tag{4.33}
\]

where the right hand side is the number of particles in the trap expressed in terms of the radius \( R_0 \) and the trap frequency \( \omega_0 \).

Fig. 4.3 shows a plot of the dimensionless cloud radius \( R/R_0 \) as a function of the dimensionless gauge coupling strength \( \lambda/k_F^0 \) of the EO gauge field. Quite remarkably, the cloud shrinks on application of the gauge field. This is consistent with the results of ref. [101]. Further, we find here that there is a critical gauge coupling \( \lambda_c \) given by

\[
\lambda_c = \left( \frac{5}{2} \right)^{1/6} k_F^0, \tag{4.34}
\]

such that when \( \lambda > \lambda_c \), the radius is given by

\[
\frac{R}{R_0} = \left( \frac{5\sqrt{2} k_F^0}{16} \lambda \right)^{1/5} \quad \text{for } \lambda \geq \lambda_c. \tag{4.35}
\]

This is the regime of gauge coupling that has the most interesting physics. This is because for \( \lambda > \lambda_c \), the local Fermi sea everywhere in the trap will contain only + helicity eigenstates. In the presence of attractive interaction between fermions in the singlet channel, a rashbon BEC is obtained at the center of the trap at low temperatures when \( \lambda \gtrsim \lambda_c \).
4.3. SIZES AND SHAPES OF TRAPPED CLOUDS

For the S GFC, briefly, the cloud radius satisfies

\[
\left( \frac{R}{R_0} \right)^6 + 2 \left( \frac{\lambda}{k_F^0} \right)^2 \left( \frac{R}{R_0} \right)^4 = 1.
\]

Indeed, even for this GFC, the cloud shrinks under the influence of the gauge field. As is evident, for \( \lambda/k_F^0 \gtrsim 1 \), the cloud radius goes as \( \frac{R}{R_0} \sim \sqrt{\frac{k_F^0}{\lambda}} \).

The discussion above makes it clear that the shrinking of the cloud is a characteristic feature obtained on application of a generic non-Abelian gauge field. It is useful to further investigate the physics behind this remarkable and readily observable effect. Let us briefly recap the factors that determine the cloud size. In a Fermi system, the “confining walls” of the container resist the Pauli (degeneracy) pressure. In this case equivalent to the confining wall is the harmonic trapping potential. It is apparent that on application of the gauge field the degeneracy pressure reduces, an inference that is consistent with the reduction of the cloud size.

Why should the Pauli pressure of the system reduce in the presence of a non-Abelian gauge field? To answer this question, we consider a homogeneous system (no trap) and obtain the stress tensor from the momentum balance equation. For our noninteracting system, the stress tensor operator is (sum over repeated indices is implied)

\[
S_{ij} = \frac{1}{8\pi^3} \int d^3k \Psi_{\sigma}^\dagger(k) k^i v^{j\sigma}(k) \Psi_{\sigma'}(k),
\]

where \( \Psi_{\sigma}^\dagger(k) \) is the fermion creation operator in \( k \) space,

\[
v^{j\sigma}(k) = \left( k^j \delta_{\sigma\sigma'} - \lambda^{(j)} \tau^{(j)}_{\sigma\sigma'} \right)
\]

are the components of the velocity operator (here repeated \( j \) is not summed). Note that the velocity operator does not commute with the Hamiltonian in eqn. (4.1). A straightforward calculation now shows that the stress tensor in our case is

\[
\langle S^{ij} \rangle = \frac{1}{8\pi^3} \int d^3k \left( \sum_{\alpha} k^i (v^j(k))_{\alpha\alpha} n_F(\varepsilon_\alpha(k) - \mu) \right),
\]
4.3. SIZES AND SHAPES OF TRAPPED CLOUDS

Figure 4.4: (Color online) Dependence of the pressure $P$ of a homogeneous gas of fermions in an EO gauge field on the strength of the gauge coupling $\lambda$. $P_0$ is the pressure of the free Fermi gas (no gauge field), and $k_F$ is the Fermi wave vector determined by the density.

where $n_F$ is the Fermi function, $\mu$ is the chemical potential and $\langle v^i(k)\rangle_{\alpha}$ is the expectation value of the velocity operator (eqn. (4.38)) in the helicity eigenstate $|k\alpha\rangle$ given in eqn. (4.2). The pressure $P$ is now obtained as

$$P = \frac{1}{3}\langle S^{ii}\rangle.$$  \hspace{1cm} (4.40)

For the EO gauge field,

$$\langle v(k)\rangle_{\alpha} = \left(\frac{k - \alpha\lambda}{\sqrt{2}}\right)e_k + k_z e_z,$$  \hspace{1cm} (4.41)

where we have used cylindrical polar coordinates in $k$ space. This formula illustrates a key point. For the positive helicity states ($\alpha = +1$), which are of lower energy, the expectation value of velocity vanishes for $k = (\lambda/\sqrt{2}, 0, 0)$. This is the result of the fact that although the canonical momentum of the low energy states is of the order of $\lambda$, the mechanical momentum which is proportional to the velocity is small. It is the mechanical momentum that contributes to the stress tensor. The reason for the fall in pressure is now evident. In the presence of the gauge field, all the states with $+\) helicity have a smaller velocity (mechanical momen-
Since in the presence of the gauge field, there are always more occupied states with + helicity, the pressure is expected to fall with increasing \( \lambda \) for a given density \( \rho \) of particles.

Fig. 4.4 illustrates the dependence of the pressure \( P \) on the gauge coupling \( \lambda \) for the EO gauge field calculated using eqn. (4.39) and eqn. (4.40). The pressure falls as expected. In fact, for \( \lambda > \lambda_T \) (\( \lambda_T = \left( \frac{8\sqrt{2}}{3\pi} \right)^{1/3} k_F \), the gauge coupling at which the Fermi surface undergoes a topological transition, \( k_F = (3\pi^2 \rho)^{1/3} \) and \( \rho \) is the density), we obtain

\[
\frac{P}{P_0} = \frac{5\sqrt{2}}{3\pi} \frac{k_F}{\lambda} \quad \text{for} \quad \lambda \geq \lambda_T.
\]

We also note that the stress tensor calculated via eqn. (4.39) is isotropic for this GFC.

Similar physics is at play also in the spherical gauge field where \( \frac{P}{P_0} \sim (\frac{k_F}{\lambda})^4 \) for gauge couplings larger than that required to produce the change in topology of the Fermi surface. The more rapid fall of pressure with increasing gauge coupling owes to the fact that there are many more low energy states with vanishing velocity in the spherical gauge field.

These considerations now provide a clear physical picture of the shrinking of the cloud with increasing gauge coupling.

### 4.3.2 Cloud Shape

Within LDA the sole effect of the gauge field is the shrinking of the cloud and cloud shape remains spherical for this isotropic trap. For the EO gauge field, this is a somewhat surprising result since one would expect the one body wave function to have different behaviors “in-plane” of the gauge field (\( x-y \) plane) and “out of plane” (\( z \) direction). To investigate this, we calculated the density profile using the exact wave functions for this problem obtained numerically using the method given in subsection of 4.2.2 and compared this with the LDA result.

Fig. 4.5 shows the density profiles obtained using the exact numerical calculation of subsection 4.2.2 and that obtained from LDA for three values of gauge coupling \( \lambda \) for the EO gauge field. For this gauge field, the symmetry of the prob-
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Figure 4.5: (Color online) Density profile of a gas of 910 trapped fermions in an EO gauge field. The in-plane density profile is indicated by $\rho(r, 0)$ and out of plane profile by $\rho(0, z)$.

The density ensures that the density is a function only of the in-plane radial coordinate $r$ and the $z$ coordinate, i.e., $\rho = \rho(r, z)$. We show curves corresponding to in-plane density profile $\rho(r, 0)$ and out of plane profile $\rho(0, z)$ obtained from the exact numerical calculation. The results show a remarkable feature. For $\lambda = 8\sqrt{2\omega_0}$, we see that in-plane and out of plane density profiles are close to each other and agree quite well with the LDA result. However, at a larger value of the gauge coupling $\lambda = 15\sqrt{2\omega_0}, 20\sqrt{2\omega_0}$, we see that the in-plane density agrees quite well
4.3. SIZES AND SHAPES OF TRAPPED CLOUDS

Figure 4.6: (Color online) Contour plot of the spatial density profile of a trapped gas of 910 fermions in an EO GFC with $\lambda = 20 \sqrt{2} \omega_0$. This clearly illustrates the anisotropy in the cloud shape induced by the gauge field.

with the LDA result. However, in both of these cases, the out of plane density systematically deviates from the in-plane (and LDA) results. Incidentally, the change in the density can also be interpreted as a result of the corresponding change in the density of states of the system upon application of the non-Abelian gauge field. Such an argument has been successfully used in explaining the increase in $T_c$ of BCS pairing under such a non-Abelian gauge field [102].

This effect can be understood by using the one particle spectrum (eqn. (4.20)), obtained using our adiabatic theory. Filling up of the levels associated with the quantum numbers $(n, m, n_z)$ occurs by the ordering of their energies. When $\lambda$ is large, the levels of the type $(0, m, 0)$ are the lowest lying ones with the spacing between them approximately equal to $2m \frac{\omega_0^2}{\lambda^2}$. On the other hand, occupation of a state with $n_z = 1$ or $n = 1$, requires an energy of order $\omega_0$ higher than the ground state energy. Thus the number of states with $n = n_z = 0$ will be determined by the maximum value of $m$ attained, $m^{\max} \sim 2\lambda/\sqrt{\omega_0}$. If $N_p$ is the number of par-
articles, we can estimate the maximum value of \( n_z \) of all occupied states. A simple calculation shows that \( n_z^{\text{max}} \approx \sqrt{\frac{\sqrt{\omega_0 N_p}}{\lambda}} \). For the smallest value of \( \lambda \) shown in Fig. 4.5, the value of \( m_r^{\text{max}}, n_z^{\text{max}} \) etc. are all approximately equal and the cloud is approximately isotropic with reasonable agreement with LDA. However, for the higher values of \( \lambda \) in Fig. 4.5, \( n_z^{\text{max}} \approx 6 \) but \( m_r^{\text{max}} \) is much larger. The reason why \( \rho(r,0) \) agrees with LDA is now evident. Contributions to \( \rho(r,0) \) arises from many states with different \( m \) values and therefore the averaging/smoothing of the density is more prominent. This is not the case for \( \rho(0,z) \) where the density contribution is obtained only using \( n_z^{\text{max}} \) number of \( z \) wave functions. Not surprisingly, the out of plane density profile \( \rho(0,z) \) shows large deviations from the LDA result. Finally, we address the reason of why \( \rho(r,0) \) falls below \( \rho(0,z) \) for larger \( \lambda \). As \( \lambda \) increases, the in-plane wave functions are “more localized” near the origin. This is because the low energy states are predominantly constructed out of plane waves with in-plane momentum of order \( \lambda \). This means that real space wave function of the particle corresponds to that in which the particle is localized over a radial distance of order \( \frac{1}{\lambda} \). These arguments suggest: a) the trap center density (for a given number of particles) scales with \( \lambda \), b) the cloud will become more and more cigar shaped extending in the \( z \) direction as \( \lambda \) increases. Indeed, our results for the trap center density for larger \( \lambda \) do agree with this argument, and for very high value of \( \lambda \), we do find the tendency for the cloud to become cigar shaped (see fig. 4.6). Quite interestingly, the density at a given distance from the trap center can be up to 10% different along the in-plane and the out of plane directions. We believe that this systematic anisotropy should be observable and measurable in experiments.

### 4.4 Summary and Conclusions

In this chapter, we have investigated the physics of trapped noninteracting fermions in the presence of a non-Abelian gauge field that induces a generalized Rashba spin-orbit coupling. Specifically:

1. An adiabatic approximation is developed to obtain the one particle levels in a trapping potential for a generic gauge field. The effective adiabatic
4.4. SUMMARY AND CONCLUSIONS

Hamiltonian for the particle includes a gauge potential arising due to the induced connection from the internal spin degree of freedom. For the spherical gauge field this gauge potential is shown to be equivalent to the field of a monopole at the origin in the momentum space. Approximate analytic formulae for the levels are obtained for high symmetry gauge field configurations of interest. For the extreme oblate GFC these results are compared with the exact results obtained numerically and excellent agreement is demonstrated.

2. The effect of the gauge field on the size of the trapped cloud of fermions is investigated within a local density approximation. It is shown, quite generically, that the cloud shrinks with the increasing strength of the gauge field. Formulae for the dependence of the cloud size on the strength of the gauge field are obtained. The physical origins of this phenomenon are elucidated by our analysis of the stress tensor.

3. The density profile obtained using the local density approximation is compared with that obtained from an exact numerical calculation. It is shown that for the EO GFC, the density distribution is anisotropic in a characteristic way and should be observable in experiments.

We believe that the results described in this chapter will be useful to design experiments involving fermions in non-Abelian gauge field. In addition, our results also open doors to new possibilities with cold atoms that arises from using the non-Abelian gauge field in conjunction with a trapping potential or other types of potentials. In particular, we have shown that a spherical non-Abelian gauge field and a harmonic trapping potential can be used to simulate the spherical geometry quantum Hall system – laboratory-realize a magnetic monopole. Interestingly, the quantum Hall system, in this case, is realized in the momentum space.
Chapter 5

Synthetic dimensions: models and single particle physics

The importance and characteristics of synthetic gauge fields in ultracold atomic systems have been illustrated in section 1.6 and in the previous chapter 4 we described their effects in the continuum on ultracold fermions. In this chapter and in the rest of the thesis, we would now consider the effects of the synthetic gauge fields in lattice geometries. For the realization of synthetic gauge fields in the lattice geometries, the phenomenon of atom-light interaction can be exploited. The basic idea behind this can be understood by considering an atom as a two level system: a ground state $|g\rangle$ and an excited state $|e\rangle$. Consider that these two level atoms are confined in a two-dimensional square lattice geometry created using counter-propagating laser beams. The tunneling in the $y$-direction happens via usual tunneling between different potential wells. But, tunneling in the $x$-direction is frozen and can occur only via laser assisted coupling. Now, if the optical potentials along the $x$-direction is such that at even (odd) sites the atoms in the ground (excited) states are allowed to be present, then a particle tunneling from an even site to an odd site by Raman transition will acquire the phase of the running wave. Hence, by going around a plaquette of the lattice (shown in fig. 5.1), a particle will experience a magnetic field. In this way, any amount of flux between 0 to $2\pi$ can be generated but the flux produced in the optical lattice is staggered [153]. Although, by suitably engineering the phases a uniform flux per plaquette can be generated. This has its own experimental challenges and
Figure 5.1: Simulation of synthetic gauge fields on a square optical lattice. A state dependent optical lattice potential is created and the two level atoms in their ground (shown as green circles) and excited states (shown as orange circles) are loaded in the optical lattice. Atoms tunnel in the $y$-direction in the usual way due to the overlap of the site Wannier functions but tunneling along the $x$-axis is prohibited and can only be induced by lasers. Hence, the atomic wave function picks up a phase from the laser. In this way, a closed loop (shown in red) encircles a flux $\phi$ but the flux produced is staggered.

has been recently realized \cite{154, 155}. And, a recent experimental realization of synthetic gauge fields in bosonic ladders \cite{156} provide boost to this research direction. In this chapter we will discuss that using the idea of synthetic dimensions flux in a “synthetic” optical lattice can be generated in a “simpler” way.

This chapter is organized in the following way. In section 5.1, the basic idea of the synthetic dimensions is introduced. The section 5.2 contains a discussion of the formulation of the single particle problem of this system using exact diagonalization and a tight binding model for a flux of the form $2\pi \frac{L}{q}$ where $p$ and $q$ are co-prime integers with a comparison of the results of the two models. By going to a transformed basis a tight binding model for general flux with its corresponding band structure is described in section 5.3 and the following section 5.4 contains a summary of the chapter.
5.1 Synthetic dimensions

As proposed by Celi et al. [157], consider a system of atoms having $M$ hyperfine states labeled by $\gamma$ ($\gamma = 1, 2, \ldots, M$) loaded in a one dimensional optical lattice. There is usual hopping between two neighboring potential wells due to the overlap of site Wannier functions. This acts as the physical dimension whose $\gamma$-independent hopping $t$ from a site $j$ (with positions $x_j = jd$, $d$ – lattice spacing) to its neighbor preserves their internal state. Now, by using Raman transition (schematically shown in fig. 5.2), the hyperfine states of the atoms can also be coupled and a hopping in this hyperfine direction can also be induced (as shown in fig. 5.3). Hence, the hyperfine direction also acts as a “dimension” and is, therefore, called “synthetic dimension” (SD).

The states are coherently coupled by light of wave vector $k_\ell$ such that an atom in state $\gamma$ at site $j$ can “hop” to $\gamma + 1$ at site $j$ with an amplitude $\Omega_\gamma = \Omega_\gamma e^{-ik_\ell x_j}$. An atom picks up a phase factor $e^{-ik_\ell d}$ upon hopping around a plaquette $[(j, \gamma) \rightarrow (j+1, \gamma) \rightarrow (j+1, \gamma+1) \rightarrow (j, \gamma+1) \rightarrow (j, \gamma)]$, simulating an enclosed magnetic flux $\phi = k_\ell ld$ per plaquette. Choosing $k_\ell d = 2\pi \frac{p}{q}$ where $p$ and $q$ are co-prime integers, thus provides an alternate realization of the Hofstadter model with a $p/q$ flux per plaquette but in a finite square lattice strip since by construction, the system has clean edges or open boundary condition (OBC) in the synthetic dimension. There
5.1. SYNTHETIC DIMENSIONS

Figure 5.3: Schematic plot of the atoms with $M = 4$ (e.g. $^6Li$ in $F = \frac{3}{2}$ ground state manifold) loaded in a one dimensional optical lattice (red). The atoms are shown in solid blue circles and the faded blue circles show their hyperfine states. Panel-(i) corresponds to a relatively deeper optical lattice than panel-(ii). When the optical lattice is not so “deep”, the site Wannier functions has finite spread and hence an atom has finite probability of being in the vicinity of a site as well (shown in panel-(ii) by the dotted black arrows).

are chiral edge states present in the system [157]. This makes it an ideal platform to study quantum Hall physics [158]. But periodic boundary condition (PBC) in the synthetic direction can also be achieved by selectively coupling the highest and lowest hyperfine states using the Raman transition [157].

In cold atomic systems, a large number of systems having different $M$-s have already been realized. For example, fermionic systems like – Lithium ($^3Li^6$, $M = 2, 4$ [159–162]), Potassium ($^{19}K^{40}$, $M = 8, 10$ [106, 163, 164]), Francium ($^{87}Fr^{214}$, $M = 2, 4$ [165–167]), Strontium ($^{38}Sr^{87}$, $M = 10$ [168]), Dysprosium ($^{68}Dy^{161}$, $M = 22$ [169]), Ytterbium ($^{70}Yb^{171}$, $M = 2, 4$ [170–173]; $^{70}Yb^{173}$, $M = 6$ [36] ) etc. and bosonic systems like – Rubidium ($^{37}Rb^{87}$, $M = 3, 5, 7$ [174] and $^{37}Rb^{85}$, $M = 7, 9$ [174, 175]), Lithium ($^3Li^7$, $M = 3$ [176]), Potassium ($^{19}K^{39}$, $M = 3, 5$ [177]),
Sodium \((^{23}Na, M = 3, 5 [178])\), Cesium \((^{133}Cs, M = 7, 9 [179])\) etc.

Recently, there have been two experimental realizations of the non-interacting SD system for both fermionic \((^{173}Yb [180])\) and bosonic \((^{87}Rb [181])\) particles where the tunneling along the synthetic direction has been produced by using two off-resonant Raman laser beams of wavelength \(\lambda\) with an angle \(\theta\) between them. Then, the flux \(\phi\) is given by

\[
\phi = \frac{4\pi d}{\lambda} \sin \left(\frac{\theta}{2}\right).
\]

Hence, we note from the above equation that by changing the angle \(\theta\) between the two Raman laser beams the flux \(\phi\) per plaquette of the synthetic lattice can be tuned.

## 5.2 Comparison between ED and the TBM

The single particle spectrum of this non-interacting SD system can be obtained by an exact diagonalization (ED) exploiting the periodicity of the underlying optical potential to use the Bloch’s theorem and a nearest neighbor (n.n) tight binding model (TBM). In the experiments, an optical potential of finite depth is used. But, the n.n TBM gives an accurate description of the system only when the depth of the optical potential \(V_0\) is “very large” compared to the recoil energy \(E_R = \frac{\hbar^2 \pi^2}{2md^2}\) such that the site Wannier wave functions are well localized. But when the lattice is not “very deep”, due to the spreading of the site Wannier wave functions higher order tunnelings also become important and the problem loses particle hole symmetry. Thus for experimentally relevant lattice depths \(\left(\frac{V_0}{E_R} \sim 5\right)\), which is not “very deep” for the n.n TBM to be accurate, an exact solution of the problem is required. Also, in this limit, the atoms will have freedom to move inside individual potential wells. From this viewpoint, we formulate the ED and compare it with the TBM. The schematic structure of the detunings of the hyperfine states is explained in fig. 5.2. The important parameters of our problem are the depth of the optical potential \(V_0\), the two photon detuning \(\delta\), the wave vector of the Raman lasers \(k_l\) and the amplitude of the Raman coupling \(\Omega\). Each of them can be varied independently giving rise to different parameter regimes of the problem. The
5.2. COMPARISON BETWEEN ED AND THE TBM

tight binding hopping matrix element \( t \) is fixed by \( V_0 \) and the periodicity of the optical lattice \( d \). The amplitude part of the hopping along the synthetic direction for the experimental realizations \([180, 181]\) is

\[
\Omega_\gamma = \Omega f_\gamma
\]  

(5.2)

which depends on the coefficients

\[
f_\gamma = \sqrt{F(F + 1) - (F - \gamma + 1)(F - \gamma)}
\]

(5.3)

with the total spin of the particles \( F = \frac{M-1}{2} \) and \( \gamma = 1, \ldots, (M - 1) \).

5.2.1 Formulation of the ED

The one dimensional optical potential, giving rise to the physical dimension, is given by

\[
V(x) = V_0 \sin^2 \left( \frac{\pi x}{d} \right).
\]  

(5.4)

From here onwards, we will take \( d, \hbar \) and mass \( m \) of the particles to be unity.

The single particle Hamiltonian is, therefore,

\[
\mathcal{H} = -\frac{\Lambda^2}{2} \mathbb{1} + U(x),
\]

(5.5)

where \( \mathbb{1} \) is an \( M \times M \) identity matrix. \( U(x) \) is the total potential which is a sum of the state independent optical potential \( V(x) \), the detuning matrix \( \delta \) and the Raman coupling term \( w(x) \). For \( M = 3 \),

\[
\delta = \begin{pmatrix}
2\delta & 0 & 0 \\
0 & \delta & 0 \\
0 & 0 & 0
\end{pmatrix},
\]
with OBC in the synthetic direction \(w(x)\) is
\[
w(x) = \begin{pmatrix}
0 & \Omega e^{iklx} & 0 \\
\Omega e^{-iklx} & 0 & \Omega e^{iklx} \\
0 & \Omega e^{-iklx} & 0
\end{pmatrix},
\]
whereas with PBC in the synthetic direction \(w(x)\) is given by
\[
w(x) = \begin{pmatrix}
0 & \Omega e^{iklx} & \Omega e^{-iklx} \\
\Omega e^{-iklx} & 0 & \Omega e^{iklx} \\
\Omega e^{iklx} & \Omega e^{-iklx} & 0
\end{pmatrix}.
\]

For \(2\pi \frac{p}{q}\) flux per plaquette, this system has translational symmetry with \(q\)-sites per unit cell. Hence, we choose the reciprocal lattice vector \(G = \frac{2\pi}{qd}\) which implies \(k_l = pG\). Now, Bloch’s theorem is used to diagonalize the Hamiltonian (eqn. (5.5)). The Schrödinger equation gives
\[
\mathcal{H}\psi(x) = E\psi(x) .
\]
(5.6)

Then, using Bloch’s theorem we get the Bloch equation
\[
\frac{(k + nG)^2}{2} C_n(k) + \sum_m U_{n-m} C_m(k) = E_{n,k} C_n(k) ,
\]
(5.7)

where, \(C_n(k)\)-s are the Fourier coefficients of the wave function
\[
\psi_k(x) = \sum_{n=-\infty}^{\infty} C_n(k) \langle x|k + nG\rangle ,
\]
(5.8)

\(U_n\)-s are the Fourier coefficients of the potential
\[
U(x) = \sum_{n=-\infty}^{\infty} U_n e^{inGx} .
\]
(5.9)
The Fourier coefficients \( U_n \)-s for the OBC case are given by

\[
U_0 = \left( \frac{V_0}{2} \right) \frac{1}{\tilde{1}} + \delta \tilde{1}, \quad U_q = U_{-q} = -\left( \frac{V_0}{4} \right) \frac{1}{\tilde{1}}, \\
U_p = \Omega \cdot \tilde{1}^{ou}, \quad U_{-p} = \Omega \cdot \tilde{1}^{ol},
\]

where, \( \tilde{1}^{ou} \) and \( \tilde{1}^{ol} \) are upper and lower diagonal identity matrices respectively. Whereas, for the PBC case \( U_n \)-s are

\[
U_0 = \left( \frac{V_0}{2} \right) \frac{1}{\tilde{1}} + \delta \tilde{1}, \quad U_q = U_{-q} = -\left( \frac{V_0}{4} \right) \frac{1}{\tilde{1}}, \\
U_p = \Omega \cdot \tilde{1}^{pu}, \quad U_{-p} = \Omega \cdot \tilde{1}^{pl}.
\]

For \( M = 3 \),

\[
\tilde{1}^{ou} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{1}^{pu} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
\]

\[
\tilde{1}^{ol} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{1}^{pl} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

The ED calculation is performed by choosing a “large” enough cutoff for the number of reciprocal lattice vectors and then diagonalizing the Bloch Hamiltonian to obtain the single particle spectrum.

### 5.2.2 The tight binding formulation

Within the n.n tight binding approximation, PBC in the physical direction is considered and for \( 2\pi \frac{p}{q} \) flux per plaquette we take a unit cell containing \( Mq \) number of sites (with \( q \) physical sites). So, the size of a unit cell is \( l = q \) along the physical direction. These unit cells are labeled by an integer \( n \) and the sites within a unit cell are labeled by coordinates \((j, \gamma)\). The site having \( x \)-coordinate \( j \) within the \( n \)-th unit cell has actual position \( R_{n,j} \) with respect to the original lattice given by

\[
R_{n,j} = (n - 1)l + (j - 1)d.
\]
The Raman coupling coefficients at that site is

\[ \Omega_{n_j} (\gamma) = \Omega_\gamma e^{ikl_n_j} . \]  

(5.15)

The strength \( \Omega_\gamma \) is chosen to be real. The detuning of a hyperfine state labeled by \( \gamma \) is

\[ \delta(\gamma) = (\gamma - 1)\delta , \]  

(5.16)

where \( \delta \) is the two photon Raman detuning. Hence, the general TBM Hamiltonian can be written as

\[ H = \sum_{n, (j_1, \gamma_1)} t_{n_1, (j_2, \gamma_2)} (\eta) C_{n, (j_1, \gamma_1)} C_{n_1, (j_2, \gamma_2)} + \sum_{n, (j_1, \gamma_1)} \delta(\gamma) C_{n, (j_1, \gamma_1)} C_{n_1, (j_1, \gamma_1)} + h.c , \]  

(5.17)

where, \( \eta \) gives the distance between the neighbors, \( t_{n_1, (j_2, \gamma_2)} (\eta) \) is a general hopping element in the \( n \)-th unit cell from site \((j_1, \gamma_1)\) to site \((j_2, \gamma_2)\), which are \( \eta \) neighbors away. \( C \) and \( C^\dagger \) are the usual fermion annihilation and creation operators respectively. Then, we can write this Hamiltonian in Fourier space as

\[ H = -t \sum_{k, (j_1, \gamma_1)} \left[ C_{k, (j_1, \gamma_1)} C_{k, (j_1, \gamma_1)} e^{ikl} + \sum_{j} C_{k, (j_1, \gamma_1)} C_{k, (j_1, \gamma_1)} e^{ikl} + \right] \]  

\[ + \sum_{k, (j_1, \gamma_1)} \Omega_\gamma C_{k, (j_1, \gamma_1)} C_{k, (j_1, \gamma_1)} e^{ikl} + h.c . \]  

(5.18)

Here, we have used the fact that \( k_l = pG \), where \( G \) is the reciprocal lattice vector. Hence, we have \( k + k_l = k - k_l = k \) and get an \( Mq \times Mq \) matrix which is diagonalized to obtain the tight binding spectrum.

### 5.2.3 Comparison

For a general \( 2\pi \frac{p}{q} \) flux, we will now analyze the band structure of the system. With OBC in the synthetic direction, we see edge states connecting different bands but for the PBC case they are absent as expected. The hopping amplitude \( t \) is cal-
5.2. COMPARISON BETWEEN ED AND THE TBM

culated using the overlap of the Wannier functions corresponding to a 1d optical lattice. Then, we focus on the first band and define its bandwidth calculated using TBM as $W_T$ and that using ED as $W_E$. Their difference is $\Delta W = (W_T - W_E)$.

Now, we are interested in looking at the variation of the ratio $\frac{\Delta W}{W_T}$ in different parameter regimes. In fig. 5.4, we show the behavior of this ratio as a function of $\frac{V_0}{E_R}$ (fig. 5.4(a)) and $\Omega/t$ (fig. 5.4(b)) for different values flux with PBC in the synthetic direction. From fig. 5.4(a), we see that due to the increase in the localization of the site Wannier wave functions for $\frac{V_0}{E_R} \geq 10$, the difference between the TBM and ED bands almost vanishes and the ratio goes to zero, as expected. But for experimentally relevant lattice depths ($\frac{V_0}{E_R} \sim 5$), there is more than 20% difference between the TBM and the ED bandwidths. Since the TBM bandwidths are larger than the ED bandwidths, for the TBM case correlations would be more important.

For shallow optical lattice ($\frac{V_0}{E_R}$ not so “large”), the exact bands show particle-hole asymmetry which is not captured in the tight binding model, since in this regime the higher order hoppings in addition to the nearest neighbor hopping are important. From fig. 5.4(b), we note that with the change in $\Omega$ also the bandwidth difference varies significantly.

![Figure 5.4](image)

Figure 5.4: Variation of the difference in bandwidths of the first band calculated using the TBM and the ED for $M = 6$, $p = 1$ and $\delta = 0$ with PBC in the synthetic direction for different values of $q$. The ratio $\Delta W/W_T$ is shown in (a) as a function of $\frac{V_0}{E_R}$ with a fixed $\Omega/t = 1$ and in (b) as a function of $\Omega/t$ for a fixed $\frac{V_0}{E_R} = 5$. 
5.3 Tight binding model in a transformed basis

From the comparison between the bandwidths of ED and TBM in the previous section, we noted that for lattice depths \( \frac{\hbar \omega}{\kappa R} \approx 10 \) the system can essentially be modeled by a TBM. For a general flux \( k_x = \phi \) with PBC in the physical direction having \( N \) sites, we can write down a TBM for the original lattice (without any unitcell). Within this TBM, the single particle kinetic energy \( H_K \) for the SD system can be written as

\[
H_K = H_t + H_\Omega + H_\delta, \quad \text{with}
\]

\[
H_t = -t \sum_j \sum_{\gamma=1}^{M} \left( C_{j+1,\gamma}^\dagger C_{j,\gamma} + \text{h. c.} \right),
\]

\[
H_\Omega = \sum_j \sum_{\gamma=1}^{M-1} \left( \Omega_{\gamma}^j C_{j,\gamma+1}^\dagger C_{j,\gamma} + \text{h. c.} \right),
\]

\[
H_\delta = \sum_j \sum_{\gamma=1}^{M} \delta(\gamma) C_{j,\gamma}^\dagger C_{j,\gamma},
\]

where, \( C_{j,\gamma} \) and \( C_{j,\gamma}^\dagger \) are the annihilation and creation operators for particles at site \( j, \gamma \) and obey usual commutation (anti-commutation) relations for bosonic (fermionic) particles. The first term \( H_t \) (eqn. (5.20)) corresponds to the tunneling of particles along the physical direction of the optical lattice with nearest neighbor hopping \( t \). While the second term \( H_\Omega \) (eqn. (5.21)) is the tunneling produced by Raman transitions along the synthetic direction and is associated with a complex tunneling coefficient \( \Omega_{\gamma}^j \). As described earlier, the Raman couplings \( \Omega_{\gamma}^j = e^{-ik_x x_j} \Omega_{\gamma} \) depend on the positions of the physical sites and \( \Omega_{\gamma} \) depends on the particular realizations of the SD systems and for the experimental realizations [180, 181] mentioned earlier they are given by eqn. (5.2). And, the third term \( H_\delta \) (eqn. (5.22)) is associated with the change in the onsite energies due to the Raman detunings given by \( \delta(\gamma) = [\gamma - 1 - (M - 1)/2] \delta \) with \( \delta \) being the two photon Raman detuning.

Now, we define the following local gauge transformation

\[
b_{j,\gamma}^\dagger = e^{-ik_x x_j} C_{j,\gamma}^\dagger,
\]
where, \( k_{\ell\gamma} = (\gamma - 1)k_{\ell} \). In this transformed basis, the kinetic energy operator takes the “simplified” form

\[
H_K = H_t + H_\Omega + H_\delta , \quad \text{with} \quad (5.24)
\]

\[
H_t = -t \sum_j \sum_{\gamma=1}^M \left( e^{ik\ell\gamma} b_{(j+1)\gamma}^\dagger b_{j\gamma} + \text{h. c.} \right) , \quad (5.25)
\]

\[
H_\Omega = \sum_j \sum_{\gamma=1}^{M-1} \Omega_{\gamma} \left( b_{j,\gamma+1}^\dagger b_{j,\gamma} + \text{h. c.} \right) , \quad (5.26)
\]

\[
H_\delta = \sum_j \sum_{\gamma=1}^M \delta(\gamma) b_{j,\gamma}^\dagger b_{j,\gamma} . \quad (5.27)
\]

Note that now the tunneling along the synthetic direction (in eqn. (5.26)) is position independent and this has been achieved at the cost of putting a position independent phase factor in the tunneling along the physical direction (in eqn. (5.25)).

We can diagonalize the above Hamiltonian (eqn. (5.24)) in the Fourier space. To this end, we define

\[
b_{k\gamma}^\dagger = \frac{1}{\sqrt{N}} \sum_j e^{ikx_j} b_{j\gamma}^\dagger . \quad (5.28)
\]

Using these new operators \( H_K \) reduces to

\[
H_K = \sum_{k\gamma} \left[ \{ \delta(\gamma) + \varepsilon_{\gamma}(k) \} b_{k\gamma}^\dagger b_{k\gamma} + \Omega_{\gamma} \left( b_{k(\gamma+1)}^\dagger b_{k\gamma} + \text{h. c.} \right) \right] , \quad (5.29)
\]

\[
= \sum_{ka} \epsilon_a(k) b_{ka}^\dagger b_{ka} , \quad (5.30)
\]

with

\[
b_{k\gamma}^\dagger = \sum_a R_{\gamma a}(k) b_{ka}^\dagger , \quad (5.31)
\]

where, \( a = 1, 2, \ldots, M \) labels the single particle energy levels \( \epsilon_a(k) \) of the system with momentum \( k \) and \( \varepsilon_{\gamma}(k) = -2t \cos(k - k_{\ell\gamma}) \). We can rewrite eqn. (5.31) as,

\[
b_{k\gamma}^\dagger = \sum_{k_1,a_1} R_{\gamma a_1}(k, k_1) b_{k_1,a_1}^\dagger , \quad (5.32)
\]

where,

\[
R_{\gamma a_1}(k, k_1) = R_{\gamma a_1}(k) \delta_{k, k_1} . \quad (5.33)
\]
5.3. 

TIGHT BINDING MODEL IN A TRANSFORMED BASIS

Figure 5.5: Single-particle dispersion of the first band for the $M = 3$ SD system with OBC in the synthetic direction and, $\frac{\Omega}{t} = 1$, $\delta = 0$ and different values of $\phi$.

Now, we discuss an interesting issue of order of limits. For a finite $\Omega$, when $k_\ell \to 0$, we recover the results with zero flux perturbatively. But the order of the limits: $k_\ell \to 0$ and then $\Omega \to 0$ is different from the limits in opposite order. This is because the transformation (eqn. (5.23)) is a simple gauge transformation only when there is a magnetic flux which can occur only when $\Omega \neq 0$. So for the uncoupled hyperfine states, any quantity $\hat{O}$ has to be found from the calculation at finite $\Omega$ and $k_\ell$ by taking the limits in the following order: $\lim_{\Omega \to 0} \left[ \lim_{k_\ell \to 0} \hat{O}(\Omega, k_\ell) \right]$.

Finally, we analyze the band structure of the system in this transformed basis. We concentrating on the $M = 3$ SD system with OBC as an example. Non-trivial effects of flux on the band structure is shown in fig. 5.5. From this figure, we note that the lowest single particle band changes qualitatively as a function of $\phi$: as $\phi$ increases it evolves from having a single minimum at $\phi = 0$ to having three minimums at $\phi = \pi$. This phenomenon has important consequences for a system of finite density of particles. In this case, there can be a change in topology of the non-interacting Fermi surface, a so called “Lifshitz transition”, as a function of flux. This in turn results in interesting transitions between novel many body phases [182].

We further analyze the full band structure of the $M = 3$ SD system with OBC in fig. 5.6. From fig. 5.6(a), fig. 5.6(b) and fig. 5.6(c), we first note that by changing
Figure 5.6: Band structure for the $M = 3$ case with OBC in the synthetic direction and $\frac{\Omega t}{t} = 1$. Panels (a), (b) and (c) show the variation of the band structure for different values of the flux per plaquette $\frac{\phi}{2\pi} = 0.1, 0.3$ and 0.5 respectively keeping the detuning $\delta$ to be zero. Panels (d), (e) and (f) show the effect of detuning for the values of $\frac{\phi}{2\pi} = 1, 3$ and 5 respectively on the band structure keeping the flux per plaquette $\frac{\phi}{2\pi} = 0.5$ to be fixed.
flux at a fixed $\Omega$ and zero detuning, the gaps between different bands open up and there is an interesting change in their structure. In particular, there is a regime of intermediate flux ($\phi \approx 0.4$) when the lowest band becomes very flat and as a result the kinetic energy of the system can be frustrated for a “small” density of particles and this in turn gives rise to phase separation [182]. The detuning $\delta$ also has non-trivial effects and changes the band structure qualitatively. This is shown in the fig. 5.6(d), fig. 5.6(e) and fig. 5.6(f) for a fixed flux. In addition, with the OBC along the synthetic direction, there are chiral edge states [157] present in the system which were also seen in the experiments [180, 181] and with the PBC in this direction the system has beautiful single particle Hofstadter butterfly spectrum [157] with flux taken in the form $2\pi \frac{p}{q}$.

5.4 Summary

Applying a standing wave optical potential on a system of atoms having multiple hyperfine states, a 1D optical lattice is created. The hyperfine states are coupled using Raman transitions and hence can be used as an extra dimension, called synthetic dimension. The Raman couplings produced this way depends on the position of a particular site and this generates necessary position dependent Peierls phase for putting flux in an optical lattice. As a result, the system becomes a finite strip Hofstadter model with tunable flux per plaquette. When the depth of the optical lattice is very large (compared to the recoil energy), the SD system can be modeled using a nearest neighbor TBM. But for shallow optical lattice, higher order tunnelings also become important. We have formulated an ED to take this into account. For flux of the form $2\pi \frac{p}{q}$, the problem has translational symmetry with unit cells containing $q$ physical sites. We find that for experimentally relevant lattice depths the TBM and ED bandwidths differ from each other by more than 20%. In particular, for shallow optical lattice the ED bands show particle-hole asymmetry which is not captured by the nearest neighbor TBM. But for “large” enough lattice depths, we can still use a nearest neighbor TBM and we show that a new basis can be generated using a gauge transformation. In this transformed basis, the Hamiltonian of the system takes a “simplified” form. Finally, we analyze the band structure of the system in this basis.
Chapter 6

Two body physics in synthetic dimensions

The interplay of synthetic gauge fields and interaction gives rise to novel phenomena such as the generation of topological phases with exotic pairing [183, 184], new quantum phases arising solely due to the presence of synthetic gauge fields [100, 102, 185, 186] etc. Also, systems with $SU(M)$ symmetric interactions are generically interesting, both in condensed matter [187–191] and cold atoms [36, 37, 168, 192–195]. In addition, we described in the previous chapter 5 that the problem of fermions moving in the synthetic dimensions (SD) has recently received a lot of both theoretical [157, 158, 196–199] and experimental [180, 181] interests because of its ability to produce flux in an optical lattice in a “simpler” way. As has been elucidated there, the non-interacting SD system itself is very rich and the kind of systems in which the experimental realizations of the SD system is naturally possible also have $SU(M)$ symmetric interactions between the particles. Hence, being naturally motivated, we consider the SD system with $SU(M)$ symmetric interactions which is expected to have interesting physics and concentrate on the two-body physics of the system in this chapter. Previous studies [193, 200–202] with fermions in a 1d optical lattice under the influence of $SU(M)$ symmetric interactions were with no magnetic flux and without the coupling between the hyperfine states. They showed that $SU(M)$ singlet $M$-body bound states form in the system and produce quasi-long-range color superfluidity [203]. In this chapter, we analyze the two-body physics of the $SU(M)$ in-
6.1. SU(M) SYMMETRIC INTERACTION

The most natural kind of interaction present in the SD systems is an SU(M) symmetric interaction. This in terms of the original operators of the SD system de-
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fined in eqn. (5.19) is given by,

\[ H_U = -\frac{U}{2} \sum_{j,\gamma,\gamma'} C_{j\gamma}^\dagger C_{j\gamma'}^\dagger C_{j\gamma'} C_{j\gamma} . \] (6.1)

Here, \( U(>0) \) is the strength of the attractive interaction. We note that it is of contact type and two-body, i.e., it is operative only when two particles come to the same physical site. But it is “long-ranged” in the synthetic direction since any two hyperfine states in the synthetic direction can interact with each other with the same strength \( U \), e.g., two atoms at site \( j \) with \( \gamma = 1 \) and 2 will interact with the same strength as those with \( \gamma = 1 \) and \( M \). The long-ranged nature of the interaction is expected to have interesting effects on the physics of the system.

We now note that in the transformed basis (eqn. (5.23)), \( H_U \) takes the form

\[ H_U = -\frac{U}{2} \sum_{j,\gamma,\gamma'} b_{j\gamma}^\dagger b_{j\gamma'}^\dagger b_{j\gamma'} b_{j\gamma} . \] (6.2)

The SD system in this transformed basis is schematically depicted in fig. 6.1, with the notation \( k_{i,\gamma} = (\gamma - 1)\phi \). In the following section, we formulate the problem of two-particle scattering of the system in this basis.

6.2 Formulation of the two body problem

We now investigate the scattering of two particles moving in the SD system and interacting via \( H_U \) (eqn. (6.2)) using the \( T \)-matrix formulation. We consider the transformed basis (eqn. (5.23)), use PBC in the physical direction and OBC in the synthetic direction and take momentum as a good quantum number. Then in momentum space \( H_U \) (eqn. (6.2)) takes the form

\[
H_U = -\frac{U}{2N^2} \sum_{j,\gamma,\gamma'} \sum_{k_1,k_2,k_3,k_4} e^{-i(k_1+k_2-k_3-k_4)x_j} \left( b_{k_1\gamma}^\dagger b_{k_2\gamma'}^\dagger b_{k_3\gamma'} b_{k_4\gamma} \right), \\
= -\frac{U}{2N^2} \sum_{\gamma,\gamma'} \sum_{k_1,k_2,k_3,k_4} N\delta_{k_1+k_2,k_3+k_4} \left( b_{k_1\gamma}^\dagger b_{k_2\gamma'}^\dagger b_{k_3\gamma'} b_{k_4\gamma} \right), \\
= -\frac{U}{2N} \sum_{Q,\gamma,\gamma'} \left( \sum_k b_{(Q+k)\gamma}^\dagger b_{(Q-k)\gamma'}^\dagger \right) \left( \sum_{k'} b_{(Q-k')\gamma'} b_{(Q+k)\gamma} \right),
\] (6.3)
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where, the total center of mass momentum (COM) is \( Q = (k_1 + k_2) = (k_3 + k_4) \) and the relative momenta are \( k = \frac{(k_1 - k_2)}{2} \) and \( k' = \frac{(k_3 - k_4)}{2} \). We define an operator which creates a pair with COM \( Q \) and relative momentum \( k \) by

\[
P^\dagger_Q(\gamma, \gamma') = \sum_k b^\dagger_{\left(\frac{Q}{2} + k\right)\gamma} b^\dagger_{\left(\frac{Q}{2} - k\right)\gamma'} .
\]  

(6.4)

Denoting \( \sim \sum_k \) as the sum over only \( k \geq 0 \) states and \( \zeta = +1 (-1) \) for bosonic (fermionic) particles, the pair creation operator (eqn. (6.4)) can be written as,

\[
P^\dagger_Q(\gamma, \gamma') = \frac{1}{2} \sum_k \left[ b^\dagger_{\left(\frac{Q}{2} + k\right)\gamma} b^\dagger_{\left(\frac{Q}{2} - k\right)\gamma'} + \zeta b^\dagger_{\left(\frac{Q}{2} - k\right)\gamma} b^\dagger_{\left(\frac{Q}{2} + k\right)\gamma'} \right] ,
\]

\[
= \sim \sum_k \left( b^\dagger_{\left(\frac{Q}{2} + k\right)\gamma} b^\dagger_{\left(\frac{Q}{2} - k\right)\gamma'} + \zeta b^\dagger_{\left(\frac{Q}{2} - k\right)\gamma} b^\dagger_{\left(\frac{Q}{2} + k\right)\gamma'} \right) ,
\]

\[
= \sum_{k_1,a_1,k_2,a_2} \left[ \sim \sum_k \left\{ R_{\gamma a_1} \left( \frac{Q}{2} + k, k_1 \right) R_{\gamma' a_2} \left( \frac{Q}{2} - k, k_2 \right) + \zeta R_{\gamma' a_1} \left( \frac{Q}{2} + k, k_1 \right) R_{\gamma a_2} \left( \frac{Q}{2} - k, k_2 \right) \right\} \right] b^\dagger_{k_1 a_1} b^\dagger_{k_2 a_2} ,
\]

\[
= \sum_{k_1,a_1,k_2,a_2} V^Q_{\gamma,\gamma'}(k_1 a_1, k_2 a_2) b^\dagger_{k_1 a_1} b^\dagger_{k_2 a_2} ,
\]  

(6.5)

where, we have used the notation

\[
V^Q_{\gamma,\gamma'}(k_1 a_1, k_2 a_2) = \sim \sum_k \left\{ R_{\gamma a_1} \left( \frac{Q}{2} + k, k_1 \right) R_{\gamma' a_2} \left( \frac{Q}{2} - k, k_2 \right) + \zeta R_{\gamma' a_1} \left( \frac{Q}{2} + k, k_1 \right) R_{\gamma a_2} \left( \frac{Q}{2} - k, k_2 \right) \right\} .
\]  

(6.6)

With these developments, the interaction term (eqn. (6.3)) can be recast as

\[
H_U = -\frac{U}{2N} \sum_{Q,\gamma,\gamma'} P^\dagger_Q(\gamma, \gamma') P_Q(\gamma, \gamma') ,
\]

(6.7)

\[
= -\frac{U}{2N} \sum_{k_1,a_1,k_2,a_2} \sum_{k_1',a_1',k_2',a_2'} \left[ \sum_{Q,\gamma,\gamma'} V^Q_{\gamma,\gamma'}(k_1 a_1, k_2 a_2) V^{Q'}_{\gamma',\gamma'}(k_1' a_1', k_2' a_2') \right] b^\dagger_{k_1 a_1} b^\dagger_{k_2 a_2} b_{k_1' a_1'} b_{k_2' a_2'} .
\]

(6.8)

Now, we discuss two interesting limiting cases:
1. $\gamma = \gamma' :-$ For every value of $\Omega$ and $k_\ell$

$$V_{\gamma,\gamma'}^{Q}(k_1 a_1, k_2 a_2) = \sum_{k} \left[ R_{\gamma a_1} \left( \frac{Q}{2} + k, k_1 \right) R_{\gamma a_2} \left( \frac{Q}{2} - k, k_2 \right) \right] (1 + \zeta) , \quad (6.9)$$

$$= 0 \quad \text{for fermions} .$$

2. $\Omega \neq 0$, $k_\ell = 0 :-$ In this case, the $R$-matrices (eqn. (5.33)) have the following structure

$$R_{\gamma a}(k, k_1) = R_{\gamma a} \delta_{k_1} \quad (6.10)$$

where, $R_{\gamma a}$ is independent of $k$. Then, we have

$$V_{\gamma,\gamma'}^{Q}(k_1 a_1, k_2 a_2) = \sum_{k} \left( R_{\gamma a_1} R_{\gamma' a_2} + \zeta R_{\gamma' a_1} R_{\gamma a_2} \right) \delta_{Q + k, k_1} \delta_{Q - k, k_2} , \quad (6.11)$$

which in turn results in (using eqn. (6.5))

$$P_{Q}(\gamma, \gamma') = \sum_{a_1, a_2} (R_{\gamma a_1} R_{\gamma' a_2} + \zeta R_{\gamma' a_1} R_{\gamma a_2}) \sum_{k} b_{k, a_1}^\dagger b_{-k - a_2}^\dagger , \quad (6.12)$$

So, for this case the interaction term (eqn. (6.7)) takes the form

$$H_{U} = -\frac{U}{2N} \sum_{Q, \{a\}} \sum_{k, k'} \left[ \sum_{\gamma, \gamma'} \left( R_{\gamma a_1} R_{\gamma' a_2} + \zeta R_{\gamma' a_1} R_{\gamma a_2} \right) \left( R_{\gamma a_1}^* R_{\gamma' a_2}^* + \zeta R_{\gamma' a_1}^* R_{\gamma a_2}^* \right) \right]$$

$$\times b_{k + a_1}^\dagger b_{k' - a_2}^\dagger b_{-k - a_2} b_{k' + a_1} ,$$

$$= -\frac{U}{2N} \sum_{Q, a_1, a_2} P_{Q}(a_1, a_2) P_{Q}(a_1, a_2) ,$$

$$\quad (6.13)$$

using commutation (anti-commutation) relations in case of bosons (fermions).

Hence, for this case the two-body scattering potential is momentum independent.
6.2. FORMULATION OF THE TWO BODY PROBLEM

6.2.1 T-matrix formulation

With the developments in the previous section, we now analyze the two body problem using the T-matrix approach closely following [100, 185, 205]. Define a two-body state

\[ |K\rangle \equiv |k_1 a_1, k_2 a_2\rangle = b_{k_1 a_1}^\dagger b_{k_2 a_2}^\dagger |0\rangle, \quad (6.14) \]

where, \(|0\rangle\) is the vacuum state. But we note that all \(|k_1 a_1, k_2 a_2\rangle\)-s are not linearly independent since another state defined as

\[ |\tilde{K}\rangle = |k_2 a_2, k_1 a_1\rangle = \zeta |K\rangle, \quad (6.15) \]

has the same kinetic energy as \(|K\rangle\). Hence, a sum

\[ \sum_{K} = \sum_{k_1 a_1, k_2 a_2}, \]

which contains all the states, needs to be modified to contain only the linearly independent states with \(k_1 \geq k_2\) and all \(a_1, a_2\). This can be defined as

\[ \sum_{\tilde{K}} = \sum_{k_1 \geq k_2, a_1, a_2}. \quad (6.16) \]

We now rewrite the interaction term (eqn. (6.8)) in a more convenient form before using it in the T-matrix analysis in the following way

\[ H_U = -\frac{U}{2N} \sum_{Q,\gamma\gamma'} \left[ \left( \sum_K V_{\gamma\gamma'}^Q (K) |K\rangle \right) \left( \sum_{K'} V_{\gamma\gamma'}^{Q*} (K') \langle K'| \right) \right]. \quad (6.17) \]

It is noted that

\[ \sum_K V_{\gamma\gamma'}^{Q} (K) |K\rangle = \sum_{\tilde{K}} \left( V_{\gamma\gamma'}^{Q} (K) |K\rangle + V_{\gamma\gamma'}^{Q} (\tilde{K}) |\tilde{K}\rangle \right), \]

\[ = \sum_{\tilde{K}} \left( V_{\gamma\gamma'}^{Q} (K) + \zeta V_{\gamma\gamma'}^{Q} (\tilde{K}) \right) \left( \frac{1}{W_{\gamma\gamma'}^{Q}} \right) |K\rangle, \quad (6.18) \]

\[ = \sum_{\tilde{K}} W_{\gamma\gamma'}^{Q} (K) |K\rangle. \]
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\[ P_{Q}^{\dagger}(\gamma, \gamma') = \sum_{K} W_{\gamma\gamma'}^{Q}(K) |K\rangle . \]  

(6.19)

Using this in eqn. (6.17), we get

\[ H_{U} = -\frac{U}{2N} \sum_{Q,\gamma\gamma'} \sum_{K,K'} \tilde{W}_{Q}(K) W_{\gamma\gamma'}^{Q}(K') |K\rangle \langle K'| , \]

\[ = \sum_{Q} \sum_{K,K'} \tilde{W}_{Q}(K,K') |K\rangle \langle K'| , \]  

(6.20)

where, for a given \( Q \)

\[ \tilde{W}_{Q}(K,K') = \frac{U}{2N} \sum_{\gamma\gamma'} W_{\gamma\gamma'}^{Q}(K) W_{\gamma\gamma'}^{Q}(K') \equiv \frac{U}{2N} \sum_{c} W_{c}^{Q}(K) W_{c}^{Q}(K') , \]  

(6.21)

which can be thought of as an effective two body scattering potential (see eqn. (6.20)). We also note that this potential can be written as a sum of its contributions in different scattering channels \( c \equiv (\gamma\gamma') \). The total number of channels (c-s) is \( M^2 \) but only

\[ \kappa = \frac{M(M + \zeta)}{2} \]  

(6.22)

of them are truly independent because of the symmetry properties of the particles.

The two-body elastic scattering by a potential \( V \) can be described by the \( T \)-matrix equation [100, 185, 205],

\[ T = V + VG_{0}T , \]  

(6.23)

where, \( T \) denotes the \( T \)-matrix and \( G_{0} \) is the two-particle non-interacting Green’s function. Then, if the incoming particles in the state \( |K\rangle \) with total COM \( Q \) and energy \( \omega \) get scattered to the state \( |K'\rangle \) then the above equation can be written as

\[ T_{Q}(K,K';\omega) = V_{Q}(K,K') + \sum_{K_{1}} V_{Q}(K,K_{1}) G_{0}(K_{1},\omega) T_{Q}(K_{1},K';\omega) . \]  

(6.24)

For this case, the two particle non-interacting Green’s function is given by

\[ G_{0}(K,\omega) = \frac{1}{\omega^{+} - E(K)} \]  

(6.25)
with \( E(K) = [\epsilon_{a_1}(k_1) + \epsilon_{a_2}(k_2)] \) being the non-interacting two particle spectrum obtained using the single particle spectrum in eqn. (5.30) and \( \omega^+ = \omega + i0 \).

Now, we plug in the effective scattering potential (eqn. (6.21)) in the \( T \)-matrix eqn. (6.24) to obtain

\[
T_Q(K, K'; \omega) = \mathcal{W}_Q(K, K') + \sum_{K_1} \mathcal{W}_Q(K, K_1) G_0(K_1, \omega) T_Q(K_1, K'; \omega) ,
\]

\[
= - \left[ \frac{U}{2N} \sum_c W^Q_c(K) W^Q_c(K')
+ \frac{U}{2N} \sum_c W^Q_c(K) \sum_{K_1} \sim W^Q_c(K_1) G_0(K_1, \omega) T_Q(K_1, K'; \omega) \right] ,
\]

\[
= - \frac{U}{2N} \sum_c W^Q_c(K) \left[ W^Q_c(K') + \Gamma^Q_c(K', \omega) \right] ,
\]

(6.26)

where,

\[
\Gamma^Q_c(K, \omega) = \sum_{K_1} \sim W^Q_c(K_1) G_0(K_1, \omega) T_Q(K_1, K; \omega)
\]

\[
= - \frac{U}{2N} \sum_{K_1} \sim W^Q_c(K_1) G_0(K_1, \omega) \sum_{c_1} W^Q_{c_1}(K_1) \left[ W^Q_{c_1}(K) + \Gamma^Q_{c_1}(K, \omega) \right] ,
\]

[using eqn. (6.26)]

\[
= -U \sum_{c_1} \left( \frac{1}{2N} \sum_{K_1} \sim W^Q_c(K_1) G_0(K_1, \omega) W^Q_{c_1}(K_1) \right) \left[ W^Q_{c_1}(K) + \Gamma^Q_{c_1}(K, \omega) \right] ,
\]

\[
= -U \sum_{c_1} \Lambda^Q_{c,c_1}(\omega) \left[ W^Q_{c_1}(K) + \Gamma^Q_{c_1}(K, \omega) \right] .
\]

(6.27)

We can recast the above eqn. (6.27) in the following way

\[
\sum_{c_1} \left[ \delta_{c,c_1} + U \Lambda^Q_{c,c_1}(\omega) \right] \Gamma^Q_{c_1}(K, \omega) = -U \sum_{c_1} \Lambda^Q_{c,c_1}(\omega) W^Q_{c_1}(K) ,
\]

(6.28)

where, \( L^Q_{c,c_1}(\omega) \) is the \((c,c_1)\)-th element of the “secular” matrix \( \mathbb{L}_Q(\omega) \) which has dimension \( M^2 \times M^2 \). The “secular” eqn. (6.28) can be written as a matrix equation as

\[
\mathbb{L}_Q(\omega) \Gamma_Q(K, \omega) = -U \Delta_Q(\omega) W_Q(K) ,
\]

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\[ \Gamma_Q(K, \omega) = -U \cdot \Lambda_Q(\omega) \cdot W_Q(K) \, . \quad (6.29) \]

Here, \( \Gamma_Q(K, \omega) \) and \( W_Q(K) \) are column vectors whose \( c \)-th elements are \( \Gamma_Q^c(K, \omega) \) and \( W^c_Q(K) \) respectively. \( \Delta_Q(\omega) \) is a matrix whose \( (c,c_1) \)-th element is \( \Lambda^c_{c,c_1}(\omega) \).

The inverse of the matrix \( \Pi_Q(\omega) \) can be defined as

\[ \Pi_Q^{-1}(\omega) = \frac{D_Q(\omega)}{\text{Det}[\Pi_Q(\omega)]} \, , \quad (6.30) \]

with \( D_Q(\omega) \) being the matrix of the co-factors of \( \Pi_Q(\omega) \). Then from eqn. (6.29) we have

\[ \Gamma_Q(K, \omega) = -U \cdot D_Q(\omega) \cdot \Delta_Q(\omega) \cdot W_Q(K) \, . \quad (6.31) \]

Now, we want to rewrite the full \( T \)-matrix using eqn. (6.31) into eqn. (6.26) as

\[ T_Q(K, K'; \omega) = -\frac{U}{2N} \sum_{c} W^c_Q(K) \left\{ \text{Det}[\Pi_Q(\omega)] \cdot W^c_{Q^*}(K') - U \sum_{c_1,c_2} D^Q_{c,c_1} \Lambda^Q_{c_1,c_2}(\omega) W^c_{Q^*}(K') \right\} \, . \quad (6.32) \]

So, from the above equation we see that the poles of the \( T \)-matrix, which give the properties of the bound states, are given by the solutions of the equation

\[ \text{Det}[\Pi_Q(\omega)] = 0 \, . \quad (6.33) \]

And, since there are \( \kappa \) (eqn. (6.22)) independent channels, only \( \kappa \) number of distinct bound states exist in the system.

### 6.2.2 Numerical procedure

We solve the eqn. (6.33) numerically to obtain the bound state energy spectrum of the system. To this end, we first set up the secular matrix \( \Pi_Q(\omega) \). We note that,
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eqn. (6.6) can be rewritten as

\[ V_Q^c (K) = \left[ R_{\gamma a_1} (k_1) R_{\gamma a_2} (k_2) + \zeta R_{\gamma a_1} (k_1) R_{\gamma a_2} (k_2) \right] \sum_k \delta_{\frac{Q}{2}+k,k_1} \delta_{\frac{Q}{2}-k,k_2}, \quad (6.34) \]

\[ = O_c(K) \sum_k \delta_{\frac{Q}{2}+k,k_1} \delta_{\frac{Q}{2}-k,k_2}. \quad (6.35) \]

Similarly, we have

\[ V_Q^c (\tilde{K}) = \zeta O_c(K) \sum_k \delta_{\frac{Q}{2}+k,k_2} \delta_{\frac{Q}{2}-k,k_1}. \quad (6.36) \]

So, using eqn. (6.35) and eqn. (6.36) in eqn. (6.18), it follows that

\[ W_Q^c (K) = O_c(K) \sum_k \left( \delta_{\frac{Q}{2}+k,k_1} \delta_{\frac{Q}{2}-k,k_2} + \delta_{\frac{Q}{2}+k,k_2} \delta_{\frac{Q}{2}-k,k_1} \right) f_Q(k_1,k_2), \quad (6.37) \]

\[ = O_c(K) f_Q(k_1,k_2). \]

Then, we also have

\[ \Lambda_{c,c_1}^Q (\omega) = \frac{1}{2N} \sum_K \left[ \frac{O_c^*(K) G_0(K,\omega) O_{c_1}(K)}{\Theta_{c,c_1}(K,\omega)} \right] \left[ f_Q(k_1,k_2) \right]^2, \quad (6.38) \]

With these manipulations, the secular matrix \( \Lambda_Q(\omega) \) can now be explicitly constructed to find its determinant. Then the solutions of eqn. (6.33) give the energies of the bound states denoted by \( E_{bs}(Q) \).

From eqn. (6.19), it is seen that \( W_Q^c (K) \) gives the pair amplitude corresponding to the state \( \mid K \rangle \) at a particular channel \( c \). Hence, we can define the pair density of states (PDOS) \( g_{p,c_1}^c (\omega) \), which characterizes whether bound states will form in the system, corresponding to an incoming state at channel \( c \) and an outgoing state at
channel $c_1$ with energy $\omega$ as

$$
\frac{\pi}{2L} \sum_K W^Q_c(K) W^Q_{c_1}(K) \delta(\omega^+ - E(K)), \quad (6.39)
$$

$$
\frac{\pi}{2N} \sum_K O^c(K) O^c_{c_1}(K) [f_Q(k_1, k_2)]^2 \delta(\omega^+ - E(K)), \quad (6.40)
$$

Bound states can now form in the system in the regime below an energy value where the PDOS is zero. This defines the pairing threshold $E^p_{th}(Q)$ of the system as $E^p_{th}(Q) = \min_{(c,c_1)} \omega_{c,c_1}$, where $\omega_{c,c_1}$ is the lowest value of $\omega$ in a particular $(c,c_1)$ for which the PDOS, $g^p_{c,c_1}(\omega) = 0^+$. Also, there is another threshold known as the two-body threshold, which is just the minimum energy in the non-interacting two particle spectrum i.e. $E_{th}(Q) = \min_K E(K)$ and one interesting point to note is that in general $E_{th}(Q) \leq E^p_{th}(Q)$.

Now, we can define the following properties of the bound states:

1) **Binding energy**: The energies of the bound states with COM $Q$ are $E_{bs}(Q)$. Their binding energies, denoted by $E_b(Q)$, can be defined as $E_b(Q) = [E^p_{th}(Q) - E_{bs}(Q)]$, i.e. the energies of the bound states with respect to the pairing threshold.

2) **Mass**: The mass of a bound state with energy $E_{bs}(Q)$ can be defined as

$$
m^{-1}(Q) = \left. \frac{\partial^2 E_{bs}(Q_1)}{\partial Q_1^2} \right|_{Q_1 = Q}. \quad (6.41)
$$

In the following section, we discuss the properties of the bound states obtained using the $T$-matrix formulation in different parameter regimes.

### 6.3 Results and discussion

In this section, first we discuss the possible analytical results and then we go on to discuss the numerical results.

#### 6.3.1 Analytical solution for the zero flux case

In the limit of zero flux ($\phi = 0$), exact forms of the different bound state properties can be obtained. To demonstrate this, we note from eqn. (6.13) that for this case
the interaction term takes the form

$$H_U = \sum_{k,k',Q} S(k, k') |k\rangle \langle k'| ,$$  \hspace{1cm} (6.42)

where, $S(k, k') = -\frac{U}{N} v_{\beta,\beta'}$ and $v_{\beta,\beta'} = (\delta_{a_1,a'_1} \delta_{a_2,a'_2} + \zeta \delta_{a_1,a'_2} \delta_{a_2,a'_1})$ with the channels labeled by $\beta \equiv (a_1, a_2)$, states labeled by $|k\rangle \equiv |k, \beta\rangle = b^\dagger_Q k,a_1 b^\dagger_Q k,a_2 |0\rangle$ and

$$\sum_k \equiv \sum_k \sum_\beta .$$

To construct the $T$-matrix, we use an analog of eqn. (6.24) which can be written as

$$T^0(k,k';\omega) = S(k,k') + \sum_{k_1} S(k, k_1) G_0(k_1, \omega) T^0(k_1, k';\omega) ,$$  \hspace{1cm} (6.43)

$$\Rightarrow -\frac{U}{N} \left( v_{\beta,\beta'} + \Gamma^0_{\beta}(k';\omega) \right) ,$$  \hspace{1cm} (6.44)

where, $T^0(k,k';\omega)$ is the $T$-matrix for this case,

$$\Gamma^0_{\beta}(k',\omega) = \sum_{k_1} v_{\beta,\beta_1} G_0(k_1, \omega) T^0(k_1, k';\omega) ,$$  \hspace{1cm} (6.45)

$$= -U \sum_{\beta_1} v_{\beta,\beta_1} \Theta^0_{\beta_1}(\omega) (v_{\beta_1,\beta'} + \Gamma^0_{\beta_1}(k',\omega)) ,$$  \hspace{1cm} (6.46)

$$\Rightarrow \sum_{\beta_1} [\delta_{\beta,\beta_1} + U v_{\beta,\beta_1} \Theta^0_{\beta_1}(\omega) \Gamma^0_{\beta_1}(k',\omega) = -U \sum_{\beta_1} v_{\beta,\beta_1} \Theta^0_{\beta_1}(\omega) v_{\beta_1,\beta'} ,$$  \hspace{1cm} (6.47)

with $\Theta^0_{\beta}(\omega) = \frac{1}{N} \sum_k G_0(k, \omega)$. The above equation is analogous to eqn. (6.28) and the elements of the secular matrix $L^0_{\beta,\beta_1}(\omega)$ are given by $L^0_{\beta,\beta_1}(\omega) = [\delta_{\beta,\beta_1} + \Lambda^0_{\beta,\beta_1}(\omega)]$.

The two particle spectrum for this case is

$$\epsilon^0_Q(k) = -4t \cos(k) \cos\left(\frac{Q}{2}\right) + \Omega X_{a_1,a_2} ,$$  \hspace{1cm} (6.48)

with the functions $X_{a_1,a_2}$ given by

$$X_{a_1,a_2} = -2 \left[ \cos\left(\frac{\pi a_1}{M+1}\right) + \cos\left(\frac{\pi a_2}{M+1}\right) \right] \text{ for the special case of } \Omega_\gamma = \Omega ,$$

$$= [(2a_1 - M - 1) + (2a_2 - M - 1)] \text{ for } \Omega_\gamma \text{ given by eqn. (5.2)} .$$
Then using the following integral
\[
\int_0^\pi \frac{dk}{a + b \cos(k)} = \frac{\pi}{\sqrt{a^2 - b^2}}
\] (6.49)
we have
\[
\Theta_0^\beta(\omega^+) = \frac{1}{2} \frac{1}{\sqrt{(\omega^+ - \Omega X_{a_1,a_2})^2 - 16t^2 \cos^2 \left(\frac{Q}{2}\right)}}.
\] (6.50)

Now, exploiting the structure of \( v_{\beta,\beta_1} \), we obtain an exact analytical form of the determinant of the secular matrix to be
\[
\text{Det} \left[ \mathbb{L}_{0}^{0 \ Q}(\omega^+) \right] = \prod_{a_1,a_2} \left[ 1 - \frac{U}{\sqrt{(\omega^+ - \Omega X_{a_1,a_2})^2 - 16t^2 \cos^2 \left(\frac{Q}{2}\right)}} \right],
\] (6.51)
where, the * symbol denotes > (⩾) for fermions (bosons).

We label the bound states by \( \alpha(a_1, a_2) = 1, 2, ..., \kappa \) where, \( \kappa \) is defined in eqn. (6.22). Then, the energy of the \( \alpha \)-th bound state (given by the solutions of the equation: \( \text{Det} \left[ \mathbb{L}_{0}^{0 \ Q}(\omega^+) \right] = 0 \)) is
\[
E_{\alpha}^{bs} = \left[ -\sqrt{U^2 + 16t^2 \cos^2 \left(\frac{Q}{2}\right)} + \Omega X_{\alpha(a_1,a_2)} \right].
\] (6.52)

The PDOS is given by
\[
g_{\beta,\beta_1}^{\rho_0}(Q, \omega^+) = \text{Im} \left[ \Lambda_{\beta,\beta_1}^{0}(Q, \omega^+) \right] = \frac{\pi}{2} \frac{v_{\beta,\beta_1}}{\sqrt{16t^2 \cos^2 \left(\frac{Q}{2}\right) - (\omega^+ - \Omega X_{a_1,a_2})^2}}
\] (6.53)
and the pairing threshold is
\[
\varepsilon_{th}^{\rho_0} = -4t \cos \left(\frac{Q}{2}\right) + \Omega X_{\alpha_c(a_1,a_2)} ,
\] (6.54)
with \( \alpha_c(a_1, a_2) = (1, 2) \) for fermions and \( (1, 1) \) for bosons. The mass of the bound states, independent of \( \Omega \), has simplified form for \( Q = 0 \) given by
\[
m = \frac{\sqrt{U^2 + 16t^2}}{4t^2} ,
\] (6.55)
and hence for \( U \gg t, m \sim \frac{U}{4t^2} \) which can be explained by noting that in this limit.
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![Graphs showing energy levels and binding energies](image)

Figure 6.2: Properties of the bound states for $M = 3$ and zero flux. a) Variation of the energy of the bound states with COM $Q$ for a fixed $\Omega/t = 0.5$ and $U/t = 5$. b) Variation of the binding energy ($E_b$) with $\Omega$.

Particles hop to their neighboring sites via virtual processes with a kinetic energy gain $\sim 4t^2/U$.

As the value of $U$ decreases from a large value, the bound states can move into the scattering continuum and then a critical value of $U = U_c$ is needed for bound state formation. The expression of $U_c$ for this case, is given by

$$U_c = \sqrt{4t \cos \left( \frac{Q}{2} \right) + \Omega \left( X_{\alpha(a_1,a_2)} - X_{\alpha(c_1,c_2)} \right)^2 - 16t^2 \cos^2 \left( \frac{Q}{2} \right)}.$$  \hspace{1cm} (6.56)

It is to be noted that this is in contrary to one dimensional systems in continuum where an infinitesimal amount of attractive interaction produces bound states. Hence, the fact that even for this 1D system there is a critical value of attractive interaction above which only the bound states can form in the system is very interesting and is a result of the Zeeman field mitigating the bound state formation.

We show the variations of the bound state energies and corresponding binding energies at $Q = 0$ for the $M = 3$ case in fig. 6.2 as an example.

6.3.2 Finite flux

We noticed in the previous chapter 5 that the single particle SD system, with finite flux, itself is very rich. Naturally, the interacting SD system with finite flux gives rise to qualitatively different physics. From here on, the results of the fermionic case will be discussed, although similar analysis for bosons can be read-
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Figure 6.3: (Color online) Formation of a two-body resonance for the $M = 2$ case with $\Omega/t = 1$. (a) Shows the behaviors of the pairing threshold ($E_{p}^\text{th}$) as a function of $Q$ for different values of flux ($\phi$). We note from (a) that $E_{bs}(Q)$ of $\phi = 0$ (which is with $U/t = 4$) can be just above $E_{p}^\text{th}(Q)$ of $\phi \neq 0$ in some regime of $Q$ (hatched lines below the black curve). In this situation, the fate of the bound states is explored in (b) and (c) focusing at $Q = 0$. (b) Shows the total PDOS ($g_p(\omega)$) as a function of energy ($\omega$) close to the two-body threshold ($\omega_{th}$) for different values of $\phi$. It becomes finite as soon as $\phi \neq 0$ and increases with increasing $\phi$. (c) The phase shift ($\delta$) as a function of energy in that regime for different $\phi$. The bound state for the $\phi = 0$ case becomes resonant for $\phi \neq 0$ by picking up a finite lifetime and hence a smooth but large change in $\delta$.

ily adopted. In the following, we discuss the novel consequences of the finite flux:

- **Formation of resonances:** In the absence of any flux, Pauli blocking prevents two particles in the same $a$-states to scatter since there is no pairing amplitude. But since flux produces intermixing of different flavors, and as soon as $\phi$ becomes $\neq 0$ there is a non-zero pairing amplitude and two particles from the same $a$-states can now produce scattering. As a result, there is a sudden change in the $E_{p}^\text{th}$ from $E_{th}^p > E_{th}$ for the $\phi = 0$ case to $E_{th}^p = E_{th}$ for the $\phi \neq 0$ case. This is shown
in fig. 6.3(a) for the \( M = 2 \) case. It is seen that even for very small values of \( \frac{\phi}{2\pi} = 0.02 \) and 0.05, the \( E_{th}^0 \) is much below that of the \( \phi = 0 \) case. We also note that flux has another interesting effect: the minimum of the \( E_{th}^0(Q) \) shifts to a finite \( Q \) for \( \phi \neq 0 \). This has important consequences in the bound state formation which will be discussed shortly.

The discontinuity in the \( E_{th}^0 \) as a function of \( \phi \) can give rise to a situation when the \( E_{bs}(Q) \) for \( \phi = 0 \) \( (E_{bs}^0(Q)) \) is above the \( E_{th}^0(Q) \) for \( \phi \neq 0 \). In this case, an interesting phenomenon can take place in a regime of \( Q \) where \( E_{th}^0(Q) < E_{bs}^0(Q) \) (shown by the hatched lines below the black curve in fig. 6.3(a)). We look into this situation a bit more closely by considering the \( Q = 0 \) case. For this case, we see that a non-zero PDOS, which increases with increasing \( \phi \), appears near the \( E_{th} \) as shown in fig. 6.3(b). Also, the behavior of the PDOS where it just becomes non-zero is very different for the \( \phi \neq 0 \) case than that of the zero flux case for which it behaves as \( \sim 1/\sqrt{16t^2 - \omega^2} \). In this regime, if a bound state exists for the \( \phi = 0 \) case due to the absence of any PDOS, we expect this bound state to acquire a finite lifetime as soon as \( \phi \) becomes non-zero since the PDOS becomes non-zero.

We investigate this phenomenon by calculating the phase shift \( \delta(\omega) \) defined as \[ \delta(\omega) = \text{Arg}[T(\omega^+)]. \] Its behavior is shown in fig. 6.3(c) for different values of finite but small \( \phi \). We note that the true bound state of the \( \phi = 0 \) case gives a sharp theta function change in \( \delta(\omega) \). But at the moment \( \phi \neq 0 \), the nature of the bound state is no longer “true” bound state. Instead, its vestige as a bound state is manifested as a resonance like feature accompanying a smooth but large change in the \( \delta(\omega) \) \[185, 205\]. As \( \phi \) increases, the sharpness of the resonance decreases and the finite lifetime acquired by the bound state decreases which is because the PDOS also increases correspondingly. We also note that the resonances appears at energies dependent on \( Q \). Similar results are found in 3D Fermi gases with spin orbit coupling (SOC) \[185\] and systems with narrow Feshbach resonances \[71\]. Hence, to produce a true bound state even for this 1D system a critical amount of attraction strength \( (U_c) \) is required and this \( U_c \) can go to zero at a finite center of mass.

- **Finite momentum pairing:** Another pertinent feature, brought by the synthetic gauge field, is the formation of finite momentum two-body bound states. From fig. 6.4, we see that the minima of the bound state energy \( E_{bs}(Q) \) shifts to
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Figure 6.4: (Color online) Bound state energy ($E_{bs}$) as a function of $Q$ for different values of flux with $M = 2$, $\frac{\Omega}{T} = 1$ and $\frac{U}{T} = 5$. It is noted that $E_{bs}$ is minimum at a value of $Q = Q_g$ and the inset shows $Q_g \propto \phi$.

a finite $Q = Q_g$ for finite flux and $Q_g \propto \phi$ (shown in the inset). This is reminiscent of the $E_{th}^p(Q)$ being minimum at a finite $Q$ for finite flux. Also, then $E_{bs}(Q)$ is symmetric about this minima. So, in spite of the fact that the interaction term (eqn. (6.3)) is seemingly momentum conserving, the two-body bound states pick up a finite momentum. Previous studies in spin-orbit coupled Fermi gases with detuning and Zeeman field found similar results attributed to the broken Galilean invariance of the system [186]. These finite momentum bound states have interesting consequences in the many body setting as will be discussed in chapter 8.

- **Effects on other bound state properties:** Finally, we present an analysis of the effect of the synthetic gauge field on other properties of bound states. We show the variations of the bound state energy ($E_{bs}$), mass ($m$) and the binding energy ($E_b$) of the bound states (in fig. 6.5(a), fig. 6.5(b) and fig. 6.5(c) respectively) for $Q = 0$ and a larger value of $U$ than which causes resonances. We note that although $E_{bs}$ and $m$ change by small amounts, there is a large change in the $E_b$ as $\phi$ increases. Both $m$ and $E_b$ decrease with the increase in $\Omega$ for fixed $\phi$. The sudden reduction in $E_b$ (see fig. 6.5(c)) as soon as $\phi \neq 0$ is due to the discontinuity in $E_{th}^p$, as discussed earlier (see fig. 6.3(a)). Keeping $\phi$ and $U$ fixed, as $\Omega$ increases, the effective hopping parameter of the system increases and this acts against bound state
Figure 6.5: (Color online) Bound state properties as a function of flux at $Q = 0$, $\frac{U}{t} = 5$ for $M = 2$ case. Here, $E^0_{b_s}$, $E^0_b$ and $m_0$ are the bound state energy, binding energy and mass of the bound states of a 1d free Fermi gas respectively. In (a), (b) and (c) the energy of a bound state, its mass and binding energy as a function of $\phi$ are respectively shown.

formation (gives reduction of the binding energy seen in fig. 6.5(c)). But, flux promotes bound state formation enhancing $E_b$ with increasing $\phi$ at a fixed $\Omega$. Hence, there is a competition between $\Omega$ and $\phi$ in forming bound states. Although we see from fig. 6.5(b) that the mass varies non-monotonically for larger values of $\Omega$, first it decreases and then increases with the increase in flux. Another interesting phenomenon is that for a fixed $\phi$ and $U$, when $\Omega$ is increased or for a fixed $\phi$ and $\Omega$, when $U$ is decreased then the zeros of the “secular” matrix can move above the scattering threshold and appear in the next scattering continuum giving rise to bound states in-between the bands.
6.4 Summary

By using the $T$-matrix formalism, we have analyzed the two-body scattering problem in the SD system under the influence of an $SU(M)$ invariant “long-ranged” and contact type attractive interaction. The long-ranged nature of the interaction together with the magnetic flux gives rise to spectacular effects. The $T$-matrix is formulated both for bosons and fermions in a unified manner. The poles of the $T$-matrix give the properties of the bound states corresponding to the independent scattering channels and they are solved numerically for a general flux. An exact analytical solution of this two-body problem in the limiting case of zero flux is presented. Considering the fermionic case for illustration, we showed that the flux has interesting effects on the bound state spectrum of the system. In some parameter regimes, it gives rise to two-body resonances and the sharpness of the resonances decreases as flux increases giving a finite lifetime to the bound states which decreases with increasing flux. It also gives rise to finite momentum dimers which have important consequences in the many-body setting. Different properties of these bound states in different parameter regimes are also discussed.
Chapter 7

Squished baryons in synthetic dimensions

In chapter 5, we discussed that the “synthetic dimension” proposal [157] uses atoms with $M$ internal states in a 1d optical lattice, to realize a hopping Hamiltonian equivalent to the Hofstadter model (tight binding model with a given magnetic flux per plaquette) on an $M$-sites wide square lattice strip. In this chapter, we investigate few body physics of the SD system with an $SU(M)$ symmetric interaction introduced in the previous chapter 6. Previous studies [193, 200–202] of fermions with attractive $SU(M)$-interactions in a 1d lattice (no flux and Raman coupling) showed the formation of $SU(M)$ singlet “baryons” and their quasi-long-range color superfluidity [203]. We show that the SD system is equivalent to particles (with $SU(M)$ symmetric interactions) experiencing an $SU(M)$ Zeeman field at each lattice site and a non-Abelian $SU(M)$ gauge potential that affects their hopping. This equivalence brings out the possibility of generating non-local interactions between particles at different sites of the optical lattice. In addition, the gauge field induces a flavor-orbital coupling, which mitigates the “baryon breaking” effect of the Zeeman field. For $M$ particles, concomitantly, the $SU(M)$ singlet baryon which is site localized in the usual 1d optical lattice, is deformed to a non-local object (“squished baryon”). We conclusively demonstrate this effect by analytical arguments and exact (numerical) diagonalization studies. Viewed from the SD perspective, the $SU(M)$-interaction manifests as “infinite-ranged” (distance-independent) along the SD. This novel aspect taken together with the
7.1 Model and Mapping

As discussed in the chapter 5 and chapter 6, we denote the operator that creates a fermion (while we discuss the physics of fermions, many of our conclusions will be applicable also to bosons as well) at site $j$ with hyperfine flavor $\gamma$ as $C^\dagger_{j,\gamma}$. The lattice spacing $d$ of the optical lattice is taken to be unity here. We consider flux of the form $2\pi \frac{p}{q}$ where $p$ and $q$ are co-prime integers and call this SD system as $\frac{p}{q}$-flux case. The Hamiltonian of fermions moving in the synthetic dimensions and interacting via an attractive $SU(M)$ interaction is $\mathcal{H} = H_t + H_\Omega + H_U$ (taking the two photon detuning $\delta = 0$ and using eqn. (5.25), eqn. (5.26) and eqn. (6.1)), with

$$H_t = -t \sum_{j} \sum_{\gamma=1}^{M} \left( C^\dagger_{j+1,\gamma} C^\dagger_{j,\gamma} + \text{h. c.} \right),$$

$$H_\Omega = \sum_{j} \sum_{\gamma=1}^{M-1} \left( \Omega^j_{\gamma} C^\dagger_{j,\gamma+1} C_{j,\gamma} + \text{h. c.} \right),$$

$$H_U = -\frac{U}{2} \sum_{j,\gamma,\gamma'} \left( C^\dagger_{j,\gamma} C^\dagger_{j,\gamma'} C_{j,\gamma'} C_{j,\gamma} \right). \quad (7.1)$$
By using a mapping invoking a local unitary transformation, we can gain further insights into the effects of the interaction. Towards this end, we introduce the notation \( C_j = (C_{j,1}, C_{j,2}, \ldots, C_{j,M})^T \). We define a local unitary transformation
\[
C_j = \mathcal{W}_j a_j \quad \text{where,} \quad \mathcal{W}_j = \text{Diag}\{e^{-ik_{\ell_\gamma}x_j}\}, \quad \gamma = 1, \ldots, M.
\]
Here, \( k_{\ell_\gamma} = (\gamma - 1)k_{\ell} \), and \( a_j = (a_{j,1}, \ldots, a_{j,M})^T \) is another set of fermionic operators. This results in
\[
H_\Omega = \sum_j a_j^\dagger \Omega a_j \quad \text{where} \quad \Omega \text{ is a site independent Hermitian matrix given by}
\]
\[
\Omega = \begin{pmatrix}
0 & \Omega_1^* & 0 & \cdots & 0 \\
 \Omega_1 & 0 & \Omega_2^* & \cdots & 0 \\
0 & \Omega_2 & 0 & \cdots & \vdots \\
 \vdots & \vdots & \vdots & \ddots & \Omega_{M-1}^* \\
0 & 0 & \cdots & \Omega_{M-1} & 0 
\end{pmatrix}. \tag{7.2}
\]
On diagonalization \( \Omega = S\omega S^\dagger \) where \( \omega = \text{Diag}\{\omega_\zeta\} \) \( (\zeta = 1, \ldots, M) \) is a diagonal matrix with eigenvalues \( \omega_\zeta \), and \( S \) is a unitary matrix. While we choose the simplest case \( \Omega = \Omega \) to illustrate the physical ideas, our calculations can be adapted to specific systems. For this case, \( \omega_\zeta = -2\Omega \cos\left(\frac{\pi\zeta}{M+1}\right) \).

Then, \( H_\Omega = \sum_j b_j^\dagger \omega b_j \), where \( S^\dagger a_j = b_j = \{b_j,\zeta\}^T \) is a new set of fermionic operators. Clearly, \( C_j = U_j b_j \) where \( U_j = \mathcal{W}_j S \) is a unitary matrix. We now have
\[
H = -t \sum_j \left( b_{j+1}^\dagger U_{j+1}^\dagger U_j b_j + \text{h. c.}\right) + \sum_j \{b_j^\dagger \omega b_j + H_U\}, \tag{7.3}
\]
where, \( H_U \) is rewritten in terms of \( b_{j,\zeta} \) owing to its \( SU(M) \) invariance. We immediately see that in terms of the states \( b_j \), the Hamiltonian can be interpreted as that of particles in a “flavor” \( (\zeta) \) dependent potential \( \omega_\zeta \) (which is a \( SU(M) \) Zeeman field). Their hopping is influenced by an \( SU(M) \) symmetric non-Abelian gauge field \( U_{j+1}^\dagger U_j = S^\dagger \Phi S, \quad (\Phi = \text{Diag}\{e^{ik_{\ell_\gamma}d}\}) \) that produces flavor-orbital coupling. The Zeeman field depends solely on \( \Omega_\gamma \), while the gauge field has a crucial additional dependence on the flux. SD system is thus equivalent to \( SU(M) \)-interacting fermions experiencing \( SU(M) \) Zeeman and \( SU(M) \) non-Abelian gauge fields which produce flavor-orbital coupling.
7.2 Induced Interactions

We now discuss the key outcome of the mapping described in the previous section. Consider the $M = 2$ system with 2 particles. A rather unnatural limit of vanishing hopping $t \to 0$ reveals the main idea. The Zeeman field is $\omega = \text{Diag}\{-\Omega, \Omega\}$. When $\Omega \ll U$, the ground state is an $M = 2$ baryon with two particles at the same site (Fig. 7.1, top-left). The “baryon breaking” effect of the Zeeman field occurs when $\Omega$ exceeds $\Omega_c = \frac{U}{2}$ (Fig. 7.1, top-right). The broken baryon has both particles with $\zeta = 1$, located at two distinct sites. Now, consider the $\frac{1}{2}$-flux case and $t > 0$. The $\frac{1}{2}$-flux produces hopping indicated by the arrows in fig. 7.1 (bottom left) that does not conserve the $\zeta$ flavor due to the flavor-orbital coupling (gauge field). The degeneracy of the broken baryon states is lifted by the flavor-orbit coupled hopping – two particles with $\zeta = 1$ on neighboring sites can gain energy by hybridizing with the degenerate baryon state (bound along the SD). This induces a non-local attractive interaction between particles with $\zeta = 1$.

Figure 7.1: Non-local induced interaction: Top panel shows the state of two fermions when $t = 0$ with $M = 2$ which has $\omega_{\zeta=1} = -\Omega$ and $\omega_{\zeta=2} = \Omega$. Arrows in the left bottom panel show the hopping pattern when $t \neq 0$ in the presence of a $\frac{1}{2}$-flux. If the two particles are in the neighboring sites as shown, then this baryon can effectively hop on a dual lattice shown by the crosses (bottom right) by hybridizing with the degenerate baryon (vertical shaded bond), gaining kinetic energy. This produces a net attractive interaction between particles at neighboring sites with $\zeta = 1$. 
located on two neighboring sites. The outcome is a “squished baryon” state that generically has a bound state character along the synthetic and real dimension. In fig. 7.1, this state is a bound state of two particles that “resonates” between the vertical and horizontal bonds (fig. 7.1-bottom left), hopping on the “dual lattice” indicated by crosses in fig. 7.1 (bottom right). When $\Omega \gg \Omega_c$, the bound state is primarily made of particles with $\zeta = 1$ —“fully” squished baryon, a result of the attractive interaction between near-neighbor $\zeta = 1$ states proportional to $\left[ \frac{\Omega^2}{(2t-U)} - \frac{\Omega^2}{2t} \right] \sim \frac{\Omega^2 U}{3t^2}$. Indeed, longer range interactions are also similarly generated. A repulsive $U$ results in an induced non-local repulsion.

Similar physics applies to generic $M$. The key point is that the scale $\Omega_c$ and the resulting “broken baryon” state depend on the details of $\Omega_\gamma$. For a given $M$ and $\Omega_\gamma$, there are special fluxes that most effectively produce non-local binding and baryon squishing.

### 7.3 Exact Diagonalization

We have investigated few body physics of the system using numerical exact diagonalization. We consider a system of $N_q$ lattice sites with periodic boundary conditions. For a system with $M$ internal states, this provides $N_q M$ one particle states. So, for $N_p$ particles, the dimension of the resulting Hilbert space is $(\frac{N_q M}{N_p})$. We use translational symmetry, with $Q$, the center of mass momentum, as a good quantum number. If the ground state (GS) of the few body system has $Q = Q_g$, then we define the binding energy as

$$ E_b = [E_g(Q_g, U = 0) - E_g(Q_g, U)] , \quad (7.4) $$

where $E_g(Q_g, U = 0)$ is the GS energy of the same system with $U = 0$. To gain more insights into the properties of the GS, we characterize the few body bound states by computing 2 of their characteristic quantities: 1) the moment of inertia along the $x$-direction defined as

$$ I_{xx} = \frac{1}{d^2} \left( \frac{N_p}{2} \right) \langle \sum_{i_1>i_2} (\Delta x_{i_1,i_2})^2 \rangle , \quad (7.5) $$
and 2) average value for the synthetic coordinate defined as

$$\langle \zeta \rangle = \frac{1}{N_p} \left( \sum_i \zeta_i \right), \quad (7.6)$$

where i-s run over the particle labels 1, ..., $N_p$ and the relative distance $\Delta x_{i_1,i_2} = (x_{i_1} - x_{i_2})$.

Now, the following two criteria can be used to detect an $N_p$-particle few body bound state: 1) the binding energy ($E_b$) should be positive and 2) the $I_{xx}$ should be finite and insensitive to the spatial size of the system ($N_q$). In a completely unbound state, such as that obtained with $U = 0$, $I_{xx} \sim N_q^2$. On the other hand, the quantity $\langle \zeta \rangle$ provides a measure of squishing. E.g., with $N_p = M$, $\langle \zeta \rangle = \frac{(M+1)}{2}$ indicates the usual $SU(M)$-singlet baryon, while squishing is deduced from a value of $\langle \zeta \rangle < \frac{(M+1)}{2}$.

### 7.4 Results

First we consider the $t = 0$ limit which is rather unphysical as eluded in section 7.2. If we consider a few particle system with a general $M$, then as a function of $\frac{\Omega}{U}$ an interesting sequence of baryonic transitions occurs in the system. For example, consider the $M = 4$ case with 4-particles. At very small $\frac{\Omega}{U}$, a tetramer

![Figure 7.2: Binding energy ($E_b$) of 2-particles with usual 1d attractive Hubbard model. (a) The variation of $E_b$ with inverse of system size $1/N_q$ for different values of the interaction $U$. For a particular $U$, the binding energy saturates for smaller $1/N_q$ and hence larger system size. (b) By extrapolating to infinite system size limit we show the dependence of $E_b$ with $U$ and compare it with a relatively large system size $N_q = 50$. We note that they have excellent agreement.](image)


7.4. RESULTS

Figure 7.3: Phase diagram of the $M = 2$ case with two particles in the flux $(p/q)$ – $\Omega/U$ plane with $t/U = 0.1$ ($N_q = 30$). (a) and (b) show dependence of $I_{xx}$ and $\langle \zeta \rangle$ on flux $p/q$ and $\Omega/U$ respectively. Insets show the type of bound state stabilized.

Figure 7.4: $\frac{1}{2}$-flux phase diagram for $M = 2$ and 2 particles with $N_q = 30$. $E_b$, $I_{xx}$ and $\langle \zeta \rangle$ are respectively shown in (c), (d) and (e) in the $\frac{\Omega}{U}-\frac{t}{U}$ plane.
(baryon with 4 flavors) forms. It then breaks into two dimers (2 flavor baryons) and eventually to broken monomer (single flavor) states with increasing $\Omega/U$. Similarly, with $M = 6$ and 6 particles, the sequence of transitions as a function of $\Omega/U$ is from 6-baryon to $3+3$ baryons to $2+2+2$ baryons and finally to $1+1+1+1+1+1$ state. So, in this limit the system always tries to divide itself into baryons with equal number of flavors. With the presence of flux and $t$, these different intermediate baryonic transitions get stabilized. Although we discuss the results for finite system sizes, we can do a finite size scaling and extrapolate our results to infinite system size limit as illustrated in fig. 7.2.

Figure 7.5: Cuts of the $\frac{1}{2}$-flux phase diagram (fig. 7.4) for $M = 2$ and 2 particles with $N_q = 30$ along different directions. (a) Shows the dependence of $E_b$ on $\Omega/U$ for $t/U = 0.1$, while (b) shows $I_{xx}$ and $\langle \zeta \rangle$ for the same case. Vertical black dotted lines show $\Omega = \Omega_c$. Dependence of $E_b$ in (c) and $I_{xx}, \langle \zeta \rangle$ in (d) on $t/U$ at $\Omega = \Omega_c = U/2$ are shown. The dashed lines in (c) are results of analytical considerations at small and large $t/U$. 

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7.4. RESULTS

7.4.1 M = 2

We first choose the simplest case of $M = 2$ with 2 particles to discuss the results. In the absence of a flux $p/q \to 0$, the critical Zeeman field to break the baryon (dimer) is $\Omega_c = \frac{1}{2} \left( \sqrt{U^2 + 16t^2} - 4t \right)$. The full 2-body “phase diagram” in the $p/q - \Omega$ plane is shown in fig. 7.3(a) and fig. 7.3(b) considering the variation of $I_{xx}$ and $\langle \zeta \rangle$ respectively. We note that the baryon breaking transition indeed occurs at $p/q = 0$ and for larger $\Omega (> \Omega_c)$, there are no bound states. For $p/q = 1/2$ ($\frac{1}{2}$-flux), the situation is entirely different. $I_{xx}$ remains finite with the increase in $\Omega$, and $\langle \zeta \rangle$ goes to unity. The baryon evolves to become a squished baryon. The characters of the bound states are shown in the insets of fig. 7.3.

We have investigated the $\frac{1}{2}$-flux case in greater depth. Figs. 7.4(a), fig. 7.4(b) and fig. 7.4(c) clearly demonstrate that for the $\frac{1}{2}$-flux a bound state always exists (except when $t = 0$) irrespective of a large Zeeman field – a vivid example of the flavor-orbital coupling mitigating the baryon breaking effects of the Zeeman field. Figs. 7.5(a) and fig. 7.5(b) further demonstrate the squishing of the baryon by the flavor-orbital coupling. Finally, in figs. 7.5(c) and fig. 7.5(d) we discuss the case $\Omega = \Omega_c$. From analytic considerations, the binding energy of the squished baryon when $t \ll U$ is $\approx 2t$, $I_{xx} \approx \frac{1}{2}$ and $\langle \zeta \rangle \approx \frac{5}{4}$. Numerical binding energy at small $t$ is indeed in agreement, as are $I_{xx}$ and $\langle \zeta \rangle$ (fig. 7.5(c) and fig. 7.5(d)).

Figure 7.6: 3-particle phase diagram for the $M = 3$ case showing dependence of (a) $I_{xx}$, and (b) $\langle \zeta \rangle$ on flux $p/q$ and $\Omega/U$ for $t/U = 0.1$ ($N_q = 18$). Insets show the types of bound states stabilized.
Now, we consider the $M = 3$ case with 3 particles. Here, when $t = 0$, $\Omega_c = \frac{U}{\sqrt{2}}$ with a peculiar feature. Three distinct states are degenerate at $\Omega = \Omega_c$. These are the usual $M = 3$ baryons [202], a completely broken baryon with three particles at different sites (“1+1+1”), and partially broken “2+1” baryon which has two particles at a given site with $\zeta = 1$ and 2 and the third particle at a different site with $\zeta = 1$. Fig. 7.6(a) and (b) show the 3-body phase diagram in the $p/q$-$\Omega/U$ plane. Again, the squishing effect of the gauge field is clearly seen. Fig. 7.7(a) and (b) are for the case with $\frac{1}{2}$-flux ($t/U = 0.1$), which show the squishing of the baryon continuously (most rapidly near $\Omega_c$) with increase of $\Omega$. However,

Figure 7.7: Cuts of the few body phase diagram for $M = 3$ along different directions with $N_q = 18$. (a) and (b), $p/q = \frac{1}{2}$: (a) and (b) show the binding energy, and $I_{xx}$ and $\langle \zeta \rangle$ vs. $\Omega/U$ with $t/U = 0.1$ with $\frac{1}{2}$-flux. (c) and (d), $p/q = 1/3$: (c) and (d) show the same quantities as (a) and (b) for the 1/3-flux. Vertical black dotted lines show $\Omega = \Omega_c$. 
the process does not go on forever, and at a value of $\Omega$ somewhat larger than $\Omega_c$, the baryon completely breaks up. Therefore, the gauge field produced by the $\frac{1}{2}$-flux for this case is unable to entirely prevent the baryon breaking effect unlike the $M = 2$ case. Most interestingly, the situation changes completely if one introduces a $1/3$-flux. Squishing occurs smoothly (fig. 7.7(c) and (d)), and in fact, we believe, that there is a bound state for all $\Omega$ (we cannot verify this as $I_{xx}$ becomes large at large $\Omega$). Further, at $\Omega_c$, $E_b$ for small $t$ can be analytically inferred to be proportional to $t$. This is due to the hybridization between the “2+1” baryon hybridizing with a “1+1+1” aided by the $1/3$-gauge field (flavor-orbital coupling). This is, again, in excellent agreement with our numerical result.

### 7.4.3 $M = 4$

The novel aspect here is the presence of two critical Zeeman fields $\Omega_{c1}$ and $\Omega_{c2}$. When $t = 0$, the usual 4-baryon is destabilized to a state with two 2-baryons (each of which can be located at any site) at $\Omega_{c1} = \frac{2U}{\sqrt{5}}$. At $\Omega_{c2} = U$, this state is again broken into a $1+1+1+1$ state where each particle can be at any site distinct from others with $\zeta = 1$. Fig. 7.8(a) and (b) show the 4-body phase diagram in the $p/q$-$\Omega/U$ plane. A $\frac{1}{2}$-flux has a smooth change from the usual 4-baryon to a 2+2 baryon (bound state of 2-baryons) – another neat example of squishing. However, the $\frac{1}{2}$-flux is not able to mitigate the effects of the Zeeman field; near $\Omega_{c2}$ the squished

![Figure 7.8: 4-particle phase diagram for the $M = 4$ case showing dependence (a) $I_{xx}$ and (b) $\langle \zeta \rangle$ on flux $p/q$ and $\Omega/U$ for $t/U = 0.1$ obtained with $N_q = 8$. Insets show the type of bound state stabilized.](image-url)
Figure 7.9: Cuts of the 4-particle phase diagram for the $M = 4$ case obtained with $N_q = 8$. (a) and (b), $p/q = 1/4$: Panels (a) and (b) show the dependence of the binding energy, and $I_{xx}$ and $\langle \zeta \rangle$ on $\Omega/U$ with $t/U = 0.1$. (c) and (d) show same quantities as (a) and (b) for the 1/4-flux case. Vertical black dotted lines show $\Omega = \Omega_{c1}$ and $\Omega = \Omega_{c2}$ ($\Omega_{c1} < \Omega_{c2}$).

2+2 baryon is broken up [fig. 7.9(a) and (b)]. Remarkably, for a flux of 1/4, this transition is prevented [fig. 7.9(c) and (d)], and our calculations suggest bound state for any $\Omega$ (checking this requires larger computational resources). At $\Omega_{c1}$, it can be shown that the binding energy is proportional to $t^2$ (in-order to hybridize the 4-baryon and the $2 + 2$ baryon); our numerical calculations have borne this out.

**What are the general criteria required to produce squishing?** To produce squishing, the flavor-orbital coupling induced by the flux must be able to hybridize the degenerate states that occur at the critical Zeeman fields. For example, for $M = 4$, the flavor-orbital coupling with a 1/4-flux does hybridize the $2 + 2$ state with the
1 + 1 + 1 + 1 state, and hence the baryon is squished (unlike 1/2-flux). For a given $\Omega_{\gamma}$, an appropriate flux can be chosen to achieve this.

### 7.5 Summary

We show that the SD system [157] can be mapped to a system of $M$-flavor particles with $SU(M)$-interactions hopping on the lattice with an on-site $SU(M)$ Zeeman potential (due to $\Omega_{\gamma}$) along with an $SU(M)$ gauge field (due to flux $p/q$, and $\Omega_{\gamma}$) that controls their hopping. Further analysis reveals, inter alia, the gauge field induces i) a flavor-orbital coupling which mitigates the “baryon breaking” effects of the Zeeman field, and ii) a non-local interaction, i.e., interaction between particles at different physical sites. A crucial outcome is that under favourable circumstances, the $SU(M)$ singlet baryon (for $\Omega_{\gamma} = 0$), which is an object localized at a site $j$ but extended along the synthetic dimension, is transformed into an $M$-body bound state that is extended in real space (along $j$) which we dub as the “squished baryon”. This is demonstrated by analytical arguments and detailed exact diagonalization calculations. Our mapping further suggests new opportunities of using the SD system to simulate interesting models such as the $SU(M)$ random flux model [206].
Chapter 8

Many body physics in Synthetic Dimensions

As described in the previous chapters (chapter 5, chapter 6 and chapter 7), the physics of the SD system is interesting even in the single particle and few body level. In this chapter, we will now focus on the novel signatures in the many body setting of the interesting phenomena occurring at the single particle and few body level of this SD system. Previous studies [193, 200–202] in $M$ flavor fermions interacting via an SU($M$) symmetric interaction showed the formation of novel quasi-condensates of SU($M$) singlet baryons but without synthetic gauge fields. Here, we show that a plethora of novel phases get stabilized due to squishing and/or breaking of these SU($M$) singlet baryons because of an interplay between the synthetic gauge filed and the Zeeman field in this SD system (as described in chapter 7 at the few body level). We uncover the full many body phase diagram mainly using the numerical density matrix renormalization group (DMRG) [207, 208] and analytical technique such as Bosonization [209, 210]. The results presented in this chapter are based on the refs. [204] and [182].

This chapter is organized in the following way. In the section 8.1, we first introduce the DMRG simulation procedure for the SD system and show that FFLO states get stabilized even without imbalance in section 8.2 working with the transformed basis (eqn. (5.23)). In the section 8.3 and in all the other following sections, we discuss the many body physics of the SD system in the “flavor” basis (introduced in the previous chapter 7). Analytical results in several parameter regimes
are described in section 8.3. In section 8.4, we discuss the results for the zero flux case while in the following section 8.5, we concentrate on the $\pi$-flux case. The next section 8.6 contains the full many body phase diagram for the $M = 2$ case and in the final section 8.7, we summarize the results of this chapter.

### 8.1 DMRG simulation of the SD system

We use the finite system DMRG algorithm [207, 208] to simulate a fermionic SD system with a total number $N$ of particles with an open boundary condition (OBC) along the physical direction with $L$ sites. The system can, then, be viewed as a ladder with $M$ legs and $L$ rungs. So, the total number of sites in this synthetic ladder is $N_s = L \times M$. We define, the total density ($n$) of this ladder system as $n = \frac{N}{N_s}$. For the DMRG simulation, we use snaking [211] to convert this ladder system to an effective 1d system by taking into account appropriate Jordan-Wigner signs. The DMRG calculations have been performed retaining 400 truncated states per DMRG block with the maximum truncation error of $10^{-6}$ and we have used the DMRG code released within the Powder with Power Project (qti.sns.it). In the following section, we discuss that FFLO states can be stabilized even without “imbalance” in this system.

### 8.2 FFLO states without “imbalance”

In this section, we are interested in looking into the non-trivial effects brought solely by the synthetic gauge field and the consequence of the formation of the finite momentum dimers in the two body sector (described in the section 6.3.2) of the SD system. To this end, we work with a fermionic SD system in the transformed basis defined in the eqn. (5.23) with the kinetic energy given by eqn. (5.24) and interacting via the SU($M$) symmetric long-ranged attractive interaction given by eqn. (6.2). The presence of the “spin” flip term (eqn. (5.26)) reduces the symmetries of the problem only to the total occupation $n_j$ at a physical site $j$ to be conserved. But, we can define average number of particles in the $\gamma$-th leg as $\langle N_{\gamma} \rangle = \sum_j \langle n_{j,\gamma} \rangle$, with $n_{j,\gamma}$ being the number operator corresponding to the site $(j, \gamma)$ of
the ladder. We use the Raman couplings of the form given in eqn. (5.2). To explain the results, we consider the \( M = 2 \) case and focus in the regime of \( \Omega \) where there is no “population imbalance” between the two legs. Since the total number of particles in each leg is no longer a conserved quantity, we need to define the population imbalance between the two legs more carefully and the case with no population imbalance is defined by

\[
\langle N_1 \rangle = \langle N_2 \rangle.
\]

We then proceed to investigate the nature of the many body ground state by computing the pair correlation function (PCF)

\[
\mathcal{X}_{i,j} = \langle b_{i,1}^\dagger b_{i,2}^\dagger b_{j,2} b_{j,1} \rangle
\]

which gives the propensity of pair formation in the system. Quasi long-range coherence can then be deciphered from an algebraic decay in this \( \mathcal{X}_{i,j} \) with distance \( |i - j| \). We also define the pair momentum distribution function (PMDF) by the Fourier transform

\[
\mathcal{X}(k_n) = \sum_{l,m} \Theta_l(k_n) \Theta_m(k_n) \mathcal{X}_{l,m},
\]

where,

\[
\Theta_l(k_n) = \sqrt{\frac{2}{L+1}} \sin(k_n l)
\]

are the wave functions of spin-less fermions in a non-interacting 1D tight binding chain with OBC, where \( k_n \) takes on values \( k_n = \frac{\pi n}{L+1} \) with \( n = 1, \ldots, L \) and its minimum value is \( k_1 \) (\( \neq 0 \), due to the OBC).

The results of the variation of the PMDF \( \mathcal{X}(k_n) \) for different values of \( \phi \) are shown in fig. 8.1(a). A narrow peak of \( \mathcal{X}(k_n) \) at a finite value of \( k_n > k_1 \) suggests the formation of an FFLO ground state with the pairs having an FFLO momentum \( Q_F \). We note from fig. 8.1(a) that for the chosen value of \( \Omega \), the ground state for the \( \phi = 0 \) case, is not an FFLO state (\( Q_F = k_1 \)) and as \( \phi \) deviates from zero, the \( Q_F \) starts deviating from \( k_1 \). The linear scaling of the \( Q_F \) as a function of \( \phi \) is shown in the inset of fig. 8.1(a). This scaling is reminiscent of the scaling of the two-body dimer momentum \( (Q_g) \) shown in the inset of fig. 6.4. So, we see that the two-body
8.2. **FFLO STATES WITHOUT “IMBALANCE”**

![Graph](image)

Figure 8.1: (Color online) **Synthetic gauge field induced FFLO states:** Variation of the pair momentum distribution function for the $M = 2$ (fermionic) case with $\Omega/t = 1$, $L = 100$ and $n = 0.15$ for different values of $\phi$. Panel (a) shows that the peak of $\chi(k_n)$ shifts to finite value of $k_n = Q_F$ as $\phi$ becomes non-zero signaling the emergence of the FFLO states. The FFLO momentum $Q_F \propto \phi$ (shown in the inset of (a)). This has its origin in the formation of finite momentum dimers shown in fig. 6.4. Panel (b) shows that at a fixed $\phi$, the FFLO peak disappears continuously as $U$ decreases.

finite momentum dimers (shown in fig. 6.4) result in the FFLO ground states in the many body SD system.

We further analyze the properties of the FFLO ground states by investigating the behavior of the $\chi(k_n)$ at a fixed $\phi$ as a function of $U$ as shown in fig. 8.1(b).
It is noted that the strength of the FFLO peak gets suppressed strongly with decreasing $U$. Finally, with continuous decrease in $U$, the peak completely vanishes and gets transformed into a broad hump corresponding to a ground state with no quasi long-range order. Hence, there is quasi long-range coherence only above a critical value of $U$. This is similar to the usual 1D Fermi gas with Zeeman field and no spin orbit coupling [212]. We also note that this phenomenon is consistent with our discussion of the two-body problem in chapter 6 (the two-body bound states can form only above a critical value of attraction).

![Graph](image)

Figure 8.2: (Color online) Properties of the FFLO states for $M = 2$, $\Omega/t = 1$, $n = 0.15$ and $L = 100$. Panel (a) and (b) show the variations of the peak anomaly ($\mathcal{P}$) and the area ($\mathcal{A}$) under the $\mathcal{X}(k_n)$ curve as a function of $U$ for different values of $\phi$ respectively.
To quantify the suppression of the FFLO states with decreasing $U$ in a better way, we define the following two properties of the FFLO peaks as shown in fig. 8.2. 1) The peak anomaly \[ \mathcal{P} = [2\chi(Q_F) - \chi(Q_F + k_1) - \chi(Q_F - k_1)] \] can be thought to be proportional to the difference in the right and left discrete derivatives of $\chi(k_n)$ evaluated at $k_n = Q_F$. This measures the anomaly of the

Figure 8.3: (Color online) **Local and non-local correlation functions**: Parameters are $M = 2$, $\Omega/t = 1$, $\phi/(2\pi) = 0.2$, $n = 0.15$ and $L = 100$. (a) Variations of the local average density of particles $\langle n_{j,\gamma} \rangle$ in the $\gamma = 1$ and 2 legs as a function of the site number $j$ are shown for two different values of $U$, one in the regime where there are no FFLO states ($U/t = 1$) and the other where there are FFLO states ($U/t = 5$). (b) Behaviors of the non-local correlation functions $\chi_0$, $C_1$ and $C_2$ with respect to the central site at $L/2$ are shown. We note that the FFLO correlation ($\chi_0$) is the slowest to decay.
8.3 QUASI-CONDENSATION OF SQUISHED BARYONS

\( \mathcal{X}(k_n) \) at \( k_n = Q_F \) and when the peak diminishes \( \eta \) goes to zero. 2) The area (\( A \)) under the PMDF curve (shown in fig. 8.1) which gives a measure of the pairing of particles. As a function of \( U \), we show the variation of \( P \) in fig. 8.2(a) and that of \( A \) in fig. 8.2(b). We note that FFLO correlations get suppressed gradually with the decrease in \( U \) looking at the decrease in both \( P \) and \( A \). Also, this suppression is stronger for smaller values of \( \phi \), generating sharp decrease in \( P \) and \( A \) but for larger \( \phi \), \( P \) and \( A \) change smoothly.

In fig. 8.3, we show the behaviors of local and non-local correlation functions of the system in real space. The average density in each of the legs \( \langle n_{j,\gamma} \rangle \) is shown in fig. 8.3(a) as a function of the site number \( j \) and they show Friedel oscillations expected for a system with OBC [213]. From this figure, we stress the point that for the regime of parameters under consideration, there is no population imbalance in the system since \( \langle n_{j,1} \rangle = \langle n_{j,2} \rangle \). Hence, these FFLO states are different from those predicted in the imbalanced 1D Fermi gases [212–216] and are solely the effect of the synthetic gauge field present in the SD system (similar results of flow enhanced pairing are also seen in 3D Fermi gases with SOC [185]). Finally, in fig. 8.3(b) we show the following non-local correlation functions with respect to the central site at \( L/2 \): \( \mathcal{X}_0 = \mathcal{X}_{j,L/2} \) (see eqn. (8.2)), single particle correlation function at the lowest (\( \gamma = 1 \)) leg \( C_1 = \langle b_{j,1}^\dagger b_{L/2,1} \rangle \) and the highest (\( \gamma = 2 \)) leg \( C_2 = \langle b_{j,2}^\dagger b_{L/2,2} \rangle \). We note that the single particle correlations are short range but the PCF \( \mathcal{X}_0 \) shows algebraic decay with distance and is the slowest to decay. This signals the existence of a quasi long-range order [212] in the system with dominant FFLO correlations.

8.3 Quasi-condensation of squished baryons

For the rest of the chapter, we will now examine the consequences of the formation of the squished baryons in the few body sector of the SD system described in the chapter 7. As elucidated there, the squishing of the baryons, produced by a competition of the non-Abelian gauge field and the Zeeman filed is more conveniently described in a the “flavor” (\( \zeta \)) basis. From this viewpoint, from now on we would consider this flavor basis and work with the Hamiltonian given in eqn. (7.3) and use the notation therein. We define the non-Abelian gauge field,
8.3. QUASI-CONDENSATION OF SQUISHED BARYONS

\[
\omega_{\zeta=1} = -\Omega \\
\omega_{\zeta=2} = \Omega
\]

\[\begin{align*}
A_{1,1} & \quad A_{2,2} \\
A_{2,1} & \quad A_{1,2}
\end{align*}\]

Figure 8.4: (Color online) Schematic plot of the \(M = 2\) system with \(SU(2)\) gauge and Zeeman field. \(A_{i,i}\) are the flavor preserving hoppings but \(A_{1,2}\) and \(A_{2,1}\) are the flavor-orbital couplings. Note that the \(\zeta\) states at a site are Zeeman split and there is no hopping between them.

which influences hopping between the sites \(j\) and \(j + 1\), as

\[
A(j + 1, j) = \mathbb{U}_{j+1}^\dagger \mathbb{U}_j.
\] (8.5)

One interesting aspect is that \(A(j + 1, j) = A\), independent of \(j\). As examples, for the \(M = 2\) case

\[
A = e^{i\phi/2} \begin{pmatrix}
\cos(\phi/2) & i\sin(\phi/2) \\
i\sin(\phi/2) & \cos(\phi/2)
\end{pmatrix}
\] (8.6)

and for the \(M = 3\) case

\[
A = \begin{pmatrix}
e^{i\phi} \cos^2(\phi/2) & \frac{e^{2i\phi} - 1}{2\sqrt{2}} & -e^{i\phi} \sin^2(\phi/2) \\
\frac{e^{2i\phi} - 1}{2\sqrt{2}} & e^{i\phi} \cos(\phi) & \frac{e^{2i\phi} - 1}{2\sqrt{2}} \\
-e^{i\phi} \sin^2(\phi/2) & \frac{e^{2i\phi} - 1}{2\sqrt{2}} & e^{i\phi} \cos^2(\phi/2)
\end{pmatrix}.
\] (8.7)

Then, we can investigate the many-body physics of the SD system in this flavor basis by viewing it as a system of spinless fermions moving on a ladder with \(M\) legs and having \(L\) rungs (physical sites). So, as before, the ladder system has total \(N_s = M \times L\) number of sites and the filling can be defined as \(n = \frac{N}{N_s}\) with \(N\) is the total number of particles. The structure of hopping of the particles for the case of \(M = 2\) is shown in fig. 8.4. The sites are labeled by \((j, \zeta)\), where, \(j\) labels the physical sites and \(\zeta\) labels the flavors. We use numerical density matrix renormalization group (DMRG) algorithm [207, 208, 217] by taking upto 160 physical sites and analytical techniques like Bosonization [209, 210] to analyze the
problem. We note that for a general flux the system possesses only one symmetry, the total occupation at a particular physical site (total occupation of a rung). But, for $\phi = 0$ and $\pi$, there are $M$ additional symmetries in the system. Namely, $M$ additional U(1) symmetries conserving the total number of particles in the $M$ individual legs of the ladder.

### 8.3.1 Analytical considerations

We note that for any $M$, there are three energy scales $U$, $t$ and $\Omega$, and an additional parameter $\phi$. It is convenient to set $U$ as the basic scale to explore the novel aspects of the phase diagram of the system. The overall physics is then governed by the dimensionless parameters $t/U$, $\Omega/U$ and $\phi$. Analytical results are possible for $t/U \lesssim 1$ in the regimes $(\Omega/U \approx 0, \phi = 0)$, $(\Omega/U \approx 0, \phi = \pi)$, $(\Omega/U \gtrsim 1, \phi = 0)$ and $(\Omega/U \gtrsim 1, \phi = \pi)$.

The case $(\Omega/U \approx 0, \phi = 0)$ has been studied in the refs. [193, 200–202], the state is a quasi-superfluid of $SU(M)$ singlets and the lowest energy excitations are $M$-body excitations.

For the case $(\Omega/U \gtrsim 1, \phi = 0)$, the manifold of states with $\zeta = 1$ with energy $\omega_1 = -2\Omega \cos(\frac{\pi}{M+1})$ are populated and the system is a nearly free gas of fermions. Since all fermions are of flavor $\zeta = 1$, this corresponds to a ferromagnetic state. The lowest energy excitations for this state are 1-particle excitations.

The case $(\Omega/U \approx 0, \phi = \pi)$ is same as that of the $\phi = 0$ limit, when all the $\omega_\zeta = 0$ and hence $\phi$ has no effect. Once again, the system becomes a a quasi-superfluid of $SU(M)$ singlets and the lowest energy excitations are $M$-body excitations.

Finally we note that the case $(\Omega/U \gtrsim 1, \phi = \pi)$ is special: i) only $\zeta = 1$ and $\zeta = M$ states are relevant, ii) there is no mixing between $\zeta = 1$ and $\zeta = M$ states and there is an emergent $U(1) \times U(1)$ symmetry in the system. This case is investigated in more detail in section 8.5 and we show that the system is a Luther Emery fluid with dominant pairing correlations. These analytical results allow us to make the following startling conclusion. Consider the case where $\Omega/U \gg 1$. Now, starting from a ferromagnetic state at $\phi = 0$, we can obtain a state with leading pair correlations by increasing $\phi$ to $\pi$! A magnetic field through the plaquettes thus converts a ferromagnet to a quasi-superfluid.
8.4 Zero flux case

The Hamiltonian (eqn. (7.3)) of the $M = 2$ SD system for the zero flux case is equivalent to the one dimensional attractive Hubbard model in a magnetic field of strength $\Omega$. In this limit, at $t = 0$, there are dimer bound states. The energy of a dimer is $-U$ and the energy of two unbound particles in $\zeta = 1$ leg is $-2\Omega$. Hence, in this limit there is a direct transition from a “superfluid” (SF) state of dimers to the fully polarized Ferromagnetic state (i.e. all the unbound particles in the $\zeta = 1$ leg) as a function of $\Omega$ at $\Omega_c = \frac{U}{2}$. Whereas, at $\Omega = 0$, for small $\frac{U}{t}$ the state is an SF state of tightly bound pairs (i.e. a “BEC” like state). For large $\frac{U}{t}$, the binding energy of the dimers decreases and the state becomes an SF state of weakly bound pairs (i.e. a “BCS” like state). So, as a function of $\frac{U}{t}$ there is an equivalent of the “BEC” to “BCS” crossover [212] occurring in the system. Hence, at $\frac{U}{t} \neq 0$, it is expected that as a function of $\frac{\Omega}{U}$ there would be a sequence of transitions from an SF state to an FFLO state and then to a fully polarized ferromagnetic state at very large $\frac{\Omega}{U}$. Same as the previous section (section 8.2), we can characterize the FFLO state by looking at the PMDF. The basic difference between this FFLO state

![Figure 8.5: (Color online) Population imbalance created in the $M = 2$ SD system with zero flux. The average number of particles ($\langle N_{\zeta} \rangle$) in the two legs as a function of $\Omega$ with $\frac{U}{t} = 0.5$, $\phi = 0$, $L = 80$ and $n = 0.2$ are shown. We note that as $\Omega$ increases all the particles come to lowest leg above a critical value of $\Omega$ forming a ferromagnetic state.](image-url)
8.4. ZERO FLUX CASE

Figure 8.6: (Color online) Variation of the PMDF as a function of $k_n$ for the $M = 2$ case for different values of $\Omega$ with $t/U = 0.5$, $\phi = 0$, $L = 80$ and $n = 0.2$. The position of the peak at a particular value of $\Omega$ gives the total momentum ($Q_F$) of a pair. $Q_F = \pi n P$ which is shown in the inset and serves as a smoking-gun signature of the formation of an FFLO state.

from the one in the previous section is that this FFLO state is created due to the population imbalance created between the two legs labeled by $\zeta$. In fig. 8.5, we show that for the $M = 2$ case with zero flux, the average number of particles in the two legs differ from each other as $\Omega$ increases from 0 and eventually all the particles come to the lowest leg for large $\Omega$ forming a ferromagnetic state. For this case, we define the polarization $P = (\langle N_1 \rangle - \langle N_2 \rangle)/N$, where $\langle N_1 \rangle$ ($\langle N_2 \rangle$) is the average number of particles in the $\zeta = 1$ ($\zeta = 2$) leg. Then as shown in fig. 8.6, we note that the FFLO states are formed due to this population imbalance. The peak of the PMDF at a particular $\Omega$ indicates the total momentum $Q_F$ of the pairs.

We note that as $\Omega$ increases, the peak position shifts from $Q_F = k_1$ to $Q_F \neq k_1$ (like section 8.2) up to a critical value of $\Omega$ above which there is no pairing since all the particles are of same flavor ($\zeta = 1$). Also, $Q_F$ is given by $Q_F = \pi n P$, where $n$ is the total density of particles. This is shown in the inset of fig. 8.6. The fact that $Q \neq k_1$ and it is $\propto P$, act as a definitive signature of the presence of FFLO correlations in the system.
8.5. $\pi$-FLUX CASE

Figure 8.7: (Color online) Schematic plot of the $\pi$ flux case for a general $M$. Here, $\omega_\zeta$, $\zeta = 1, \ldots, M$ are the onsite energies. In the large $\Omega$ limit, particles hop in the lowest and the highest legs. The low energy sector, in this limit, is with particles effectively hopping in the lowest leg ($\zeta = 1$) and is marked by the two violet lines. Two neighboring sites (shown by the cyan boxes) in this leg can be taken to form a unit cell with odd (even) physical sites defined as $\uparrow$ ($\downarrow$) states.

8.5 $\pi$-flux case

In this section, we consider the $\phi = \pi$ case of the SD system. This case is special since now the particles have the particular hopping pattern shown in fig. 8.7. In the following, we discuss that an effective theory of particles moving only in the lowest leg can be constructed for this flux in the large $\Omega$ limit.

8.5.1 Effective theory in the large $\Omega$ limit

Due to the special hopping structure shown in fig. 8.7, there is an emergent $U(1) \times U(1)$ symmetry in the system in the limit $\frac{\Omega}{U} \gg 1$ since in this limit the only relevant states are with $\zeta = 1$ and $M$. Hence, the low energy physics of the system, in this limit, can be described by an effective theory of particles hopping only in the lowest leg. Then we can define new particle creation (annihilation) operators only in this leg ($\zeta = 1$) by $a_{j,s}^\dagger$ ($a_{j,s}$). Here, $j$ numbers the unit cells with 2 sites labeled by $s$. Without loss of generality, we can take $s = \uparrow, \downarrow$ and $\bar{s} = -s$. 
These operators are related to the original operators by
\[ b_{2j-1,1} = a_{j,\uparrow} \quad \text{and} \quad b_{2j,1} = a_{j,\downarrow}. \] (8.8)

The onsite energy in the lowest leg is \( \omega_1 = -2\Omega \cos\left(\frac{\pi}{M+1}\right) \equiv -\epsilon. \) Then, the effective Hamiltonian of the system in this limit can be written as
\[ \mathcal{H}_{\text{eff}} = H^{(0)} + H^{(1)}, \] (8.9)
where,
\[ H^{(0)} = -\epsilon \sum_{j,s} n_{j,s}, \] (8.10)
is the onsite energy term with the number operator defined as \( n_{j,s} = a_{j,s}^{\dagger}a_{j,s} \), and
\[ H^{(1)} = -\sum_{j,s} \left[ (a_{j+s,s}^{\dagger} + a_{j,s}^{\dagger}) \frac{t^2}{2\epsilon - Un_{j,s}} a_{j,s} + (a_{j,s}^{\dagger} + a_{j-s,s}^{\dagger}) \frac{t^2}{2\epsilon - Un_{j-s,s}} a_{j,s} \right]. \] (8.11)
Here, we follow the notation of \( s \) to be +1 (−1) for \( \uparrow (\downarrow) \) particles.

Noting that, \( n_{j,s} = 0 \) or 1 because of the fermionic nature of the particles, the following identity
\[ \frac{1}{(2\epsilon - Un_{j,s})} = \frac{1}{2\epsilon} \left[ 1 + \frac{Un_{j,s}}{(2\epsilon - U)} \right] \] (8.12)
holds. Using this identity, \( \mathcal{H}_{\text{eff}} \) can be recast into the form
\[ \mathcal{H}_{\text{eff}} = \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3, \] (8.13)
where,
\[ \mathcal{H}_0 = -\Omega_{\text{eff}} \sum_{j,s} n_{j,s}, \] (8.14)
\[ \mathcal{H}_1 = -t_{\text{eff}} \sum_{j,s} (a_{j+s,s}^{\dagger}a_{j,s} + \text{h. c.}), \] (8.15)
\[ \mathcal{H}_2 = -U_{\text{eff}} \sum_{j,s} (n_{j,s}n_{j,\bar{s}} + n_{j+s,s}n_{j,\bar{s}}), \] (8.16)
\[ \mathcal{H}_3 = -U_{\text{eff}} \sum_{j,s} (a_{j+s,s}^{\dagger}a_{j,s} + \text{h. c.})n_{j,\bar{s}}, \] (8.17)
with, $\Omega_{\text{eff}} = (\epsilon + \frac{t^2}{r})$ is the effective onsite energy, $t_{\text{eff}} = \frac{t^2}{2\epsilon}$ is the effective hopping amplitude and $U_{\text{eff}} = \frac{U^2}{2(2\epsilon - U)}$ is the effective interaction. We note that longer range (along the physical direction) induced interaction (see eqn. (8.16) and eqn. (8.17)) is generated in the system.

8.5.2 Bosonization of the effective model

Closely following the refs. [209, 210], we now bosonize the effective Hamiltonian in eqn. (8.13). The bosonized Hamiltonian, containing only the forward scattering terms, can be written in the following form

$$\mathcal{H}_b^0 = \sum_{\nu = \rho, \sigma} v_F (1 + \bar{g}_\nu) \sum_{q > 0} \left[ \sum_{r = R, L} y_{q,r,\nu}^\dagger y_{q,r,\nu} + x_\nu (y_{q,R,\nu}^\dagger y_{q,L,\nu}^\dagger + \text{h. c.}) \right] + \text{const.} , \quad (8.18)$$

where, $v_F = 2t_{\text{eff}} \sin(k_F)$ is the Fermi velocity and $k_F$ is the Fermi momentum of each of the species $s$. Here, $r$ labels the Right ($R$) and Left ($L$) movers, $\nu$ labels the two different sectors charge ($\rho$) and spin ($\sigma$). So, the spin-charge separation is clearly evident. The $y_{q,r,\nu}$-s are the bosonic annihilation operators, $x_\nu = \bar{g}_\nu$ and $g_{i\nu} = \frac{g_{\nu}}{v_F}$ with $i = 2$ and $4$ labeling the two types of forward scattering coupling constants. For this case, $g_{\rho} = -g_{\sigma} = \frac{f(k_F)}{\pi}$, where, $f(k_F) = V_0 + V_1 [1 + 2 \cos(k_F)]$ with $V_0 = V_1 = -U_{\text{eff}}$. Using Bogoliubov transformation [209], the effective Hamiltonian in eqn. (8.18) can be diagonalized as

$$\mathcal{H}_b^0 = \sum_\nu u_\nu \sum_{r=1,2} \sum_{q > 0} q d_{q,r,\nu}^\dagger d_{q,r,\nu} + \text{const.} , \quad (8.19)$$

where, the transformed bosonic Bogoliubov operators $d_{q,r,\nu}$ of flavors $r = 1, 2$ are defined by

$$d_{q,r,\nu} = [\cosh(\gamma_\nu) y_{q,r,\nu}^\dagger + \sinh(\gamma_\nu) y_{q,-r,\nu}^\dagger] \quad (8.20)$$

with

$$\cosh(\gamma_\nu) = \frac{1}{2} \left( \frac{1}{\sqrt{K_\nu}} + \sqrt{K_\nu} \right) , \quad (8.21)$$

$$\sinh(\gamma_\nu) = \frac{1}{2} \left( \frac{1}{\sqrt{K_\nu}} - \sqrt{K_\nu} \right) , \quad (8.22)$$
the Luttinger parameters \(K_\nu\) are given by

\[
K_\nu = \sqrt{\frac{1 + \bar{g}_{4\nu} - \bar{g}_{2\nu}}{1 + \bar{g}_{4\nu} + \bar{g}_{2\nu}}} = \frac{1}{\sqrt{1 - \nu\Lambda}},
\]

where, \(\Lambda = \frac{2U \cot \left( \frac{k_F}{2} \right)}{\pi(2\epsilon - U)}\). (8.23)

Here, we follow the notation of \(\nu\) to be +1 (-1) for \(\rho\) (\(\sigma\)) flavor particles. Their velocities are given by

\[
u_\nu = v_F \sqrt{(1 + \bar{g}_{4\nu})^2 - (\bar{g}_{2\nu})^2}.
\]

(8.24)

The diagonalized Hamiltonian (eqn. (8.20)) can be written in a more canonical form as

\[
\mathcal{H}_b^0 = \sum_\nu \frac{u_\nu}{2} \int_{-\frac{L}{2}}^{\frac{L}{2}} dr \left[ K_\nu : \Pi_\nu^2 : + \frac{1}{K_\nu} \right.
\]

(8.25)

in terms of the dual bosonic field operators \(\phi_\nu\) and \(\theta_\nu\) with \(\Pi_\nu = \partial_r \theta_\nu\). The pair \(\phi_\nu\) and \(\Pi_\nu\) are canonically conjugate variables.

We now consider the effects of the back scattering terms. We are only interested in the physics of the system away from half filling. Then, only the spin back scattering term is important since we are dealing with a system with attractive onsite interaction [210]. Then, the bosonized Hamiltonian containing this term can be written as (using eqn. (8.25))

\[
\mathcal{H}_b = \mathcal{H}_b^0 + \frac{2g_1}{(2\pi\alpha)^2} \int dr \cos(\sqrt{8\pi} \phi_\sigma(r))
\]

(8.26)

with \(\alpha \rightarrow 0^+\) being a cutoff, \(g_1\) is the coupling constant for the spin back scattering process. Here, we have,

\[
g_1 = V_0 + V_1[\cos(k_F) + \cos(2k_F) + \cos(3k_F)].
\]

(8.27)

Since \(g_1 \neq 0\) and \(K_\sigma < 1\), there is a spin gap [210] in the system.

Now, we want to investigate different properties of the bosonized Hamiltonian eqn. (8.26). The dominant correlation of the many body ground state of this Hamiltonian is the pair correlation (PC). We define, the PC function \((\Delta_b(r))\) cal-
Figure 8.8: (Color online) Comparison of the pair correlation function (eqn. (8.28)) calculated using the bosonization formula eqn. (8.29) and the bond pair correlation function (eqn. (8.30)) calculated using DMRG. The calculations are performed for the $\pi$-flux case with $L = 160$ rungs, $U/t = 0.5$, $\Omega/t = 4$ and a total of $N = 32$ number of particles. The panels (a), (b), (c) and (d) show the $M = 2$, $3$, $4$ and $5$ cases respectively. The density of particles is $n = N/LM$ and in eqn. (8.29) we use $k_F = \frac{\pi}{2}Mn$. The data for this figure is generated in collaboration and is from the ref. [182].

Calculated in the ground state of eqn. (8.26) as

$$\Delta_b(r) = \langle \hat{D}_j^\dagger \hat{D}_{j+r} \rangle , \text{with } \hat{D}_j = a_{j,\uparrow} a_{j,\downarrow} . \tag{8.28}$$

Then closely following [209], we calculate this PC function in this spin gaped system and it is given by,

$$\Delta_b(r) \sim \frac{c_1}{r^{K_\rho}} - \frac{c_2 \cos(2k_Fr)}{r^{(K_\rho + \frac{1}{2})}} , \tag{8.29}$$

where, $c_1$ and $c_2$ are positive ($> 0$) fitting parameters independent of $r$. 

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8.5.3 Comparison of the Bosonization and the DMRG results

In this section, we compare the results of the PC function calculated using the bosonization theory just discussed and the corresponding DMRG results in the large $\Omega$ limit. To compare with the bosonization result of eqn. (8.28), we define the bond-pair correlation (BPC) in the following way and calculate its expectation value with respect to the central physical site ($i_0 = L/2$) in the many body ground state using DMRG. In the same spirit of eqn. (8.28), we define the BPC as,

$$\Delta_s(r) = \langle \hat{D}_s^\dagger(i_0) \hat{D}_s(i_0 + r) \rangle ,$$

(8.30)

where, the bond pair creation operator at site $j$ is $\hat{D}_s^\dagger(j) = b_{j,1}^\dagger b_{j+1,1}^\dagger$ which creates a pair in the bond between the sites $j$ and $j + 1$ in the lowest leg. Here, $r$ is the distance the bonds and is even. In fig. 8.8, we show this comparison in the large $\Omega$ limit for different $M$-s with $\pi$ flux. We fit the DMRG data of eqn. (8.30) with the bosonization result given in eqn. (8.29), keeping $c_1$ and $c_2$ as fitting parameters. We note from the fig. 8.8 that there is an excellent agreement between them for all the $M$-s. These indicate the formation of squished baryon quasi-condensate in the SD system for the $\pi$-flux case with large $\Omega$ irrespective of the value of $M$.

8.6 Phase diagram of the $M = 2$ case

In this section, we would investigate the full many body phase diagram of the SD system, taking the $M = 2$ case as an example. To this end, we define the following two operators

$$\hat{O}_{i,h}^{(m)\dagger} = b_{i,1}^\dagger \cdots b_{i,m}^\dagger ,$$

(8.31)

$$\hat{O}_{i,s}^{(m)\dagger} = b_{i,1}^\dagger b_{i+1,1}^\dagger \cdots b_{i+m-1,1}^\dagger ,$$

(8.32)

with $m = 2, 3, \ldots, M$. The values of the local correlation functions of these operators at the central site ($i_0$) are defined as

$$f_h^{(m)} = \langle \hat{O}_{i_0,h}^{(m)\dagger} \hat{O}_{i_0,h}^{(m)} \rangle ,$$

(8.33)

$$f_s^{(m)} = \langle \hat{O}_{i_0,s}^{(m)\dagger} \hat{O}_{i_0,s}^{(m)} \rangle .$$

(8.34)
Flux squishes the baryons and aligns them towards the physical direction. The many-body phase diagram in the flux-Ω plane for the $M = 2$ case, is investigated by calculating the functions $f_h^{(2)}$ and $f_s^{(2)}$ which together can be thought to be acting as “order parameters”. They are shown in fig. 8.9(a) and fig. 8.9(b) respectively where they are normalized by their maximum value in the plane. The quantities $f_s^{(2)}$ and $f_h^{(2)}$ together characterize different phases (see Table 8.1). When $f_s^{(2)} \approx 0$ and $f_h^{(2)} \approx 1$, the state is a quasi-condensate of tightly bound pairs in the synthetic direction and hence is a dimer “superfluid” (DSF). Similarly, the state having $f_h^{(2)} \approx 0$ and $f_s^{(2)} \approx 1$ is a quasi-condensate of tightly bound pairs in the physical direction and hence is a quasi-condensate of fully squished dimers (FSDSF). If the squishing is relatively “small”, then $f_h^{(2)}$ is small but $> 0$ and $f_s^{(2)}$ is “large” but $< 1$. This state is a quasi-condensate of squished dimers (SDSF). In the single component metallic (SM) phase when all the particles are of same flavor $\zeta = 1$, $f_h^{(2)} \approx 0$ and $f_s^{(2)} = \rho(k_F)$, where $k_F = 2\pi n$ and $\rho(k_F) = \langle n_{i_0} n_{i_0+1} \rangle$ is the density-density correlation function of a free Fermi gas with density $n$ evaluated.

<table>
<thead>
<tr>
<th>Phases</th>
<th>$f_s^{(2)}$</th>
<th>$f_h^{(2)}$</th>
<th>$\langle n_{i_0,\zeta=1} \rangle$</th>
<th>$\langle n_{i_0,\zeta=2} \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DSF</td>
<td>$\approx 0$</td>
<td>$\approx 1$</td>
<td>$n$</td>
<td>$n$</td>
</tr>
<tr>
<td>“BCS”</td>
<td>$&gt; 0$</td>
<td>$&lt; 1$</td>
<td>$&gt; n$</td>
<td>$n$</td>
</tr>
<tr>
<td>SDSF</td>
<td>$&lt; 1$</td>
<td>$&gt; 0$</td>
<td>$n$</td>
<td>$\approx 0$</td>
</tr>
<tr>
<td>FSDSF</td>
<td>$\approx 1$</td>
<td>$\approx 0$</td>
<td>$n$</td>
<td>$0$</td>
</tr>
<tr>
<td>SM</td>
<td>$\rho(k_F)$</td>
<td>$\approx 0$</td>
<td>$2n$</td>
<td>$0$</td>
</tr>
<tr>
<td>FREE</td>
<td>$\rho(k_F^2)$</td>
<td>$n^2$</td>
<td>$n$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Table 8.1: Schematic table for classification of different phases of $M = 2$ using $f_h^{(2)}$ and $f_s^{(2)}$ together. The values $1$-s are in some units. Here, $i_0$ is the central site, $k_F = 2\pi n$ and $\rho(k_F) = \langle n_{i_0} n_{i_0+1} \rangle$ calculated for the free Fermi gas (FREE).
8.6. PHASE DIAGRAM OF THE $M = 2$ CASE

Figure 8.9: (Color online) Normalized $f_h^{(2)}$, $f_s^{(2)}$ and the central charge ($c$) are shown in the $\phi$-$\Omega$ plane for the $M = 2$ case in the panels (a), (b) and (c) respectively. We have used $L = 80$, $n = 0.1$ and $t/U = 0.5$. The solid lines are guides for the eye obtained from the kink in the average entanglement entropy (eqn. (8.36)). The data for this figure is generated in collaboration and is from the ref. [182].

at the central site $i_0$. In the FFLO phase (labeled by $F$ in fig. 8.9 for short-hand), $f_s^{(2)} \approx 0$ and $f_h^{(2)} < 1$; to distinguish it from a “BCS” like state, which forms for “large” $t/U$, we need to look at the behavior of the PMDF. The quasi long range coherence in a many body phase can be deciphered by looking at the correlations functions $\langle \hat{O}_{i,h}^{(m)}\hat{O}_{j,h}^{(m)} \rangle$ and $\langle \hat{O}_{i,s}^{(m)}\hat{O}_{j,s}^{(m)} \rangle$.

To understand the nature of the many body ground state, we also calculate the following properties of the ground state: i) Von-Neumann entropy and the central charge and ii) Excitation energies above the ground state.

Consider the full system to be consisted of two subsystems $A$ and $B$ of length $x$ and $(L - x)$ respectively and their corresponding Hilbert spaces are denoted by
$H_A$ and $H_B$ respectively. The total Hilbert space ($H$) of the system can then be written as a direct product $H = H_A \otimes H_B$. Then, if $\rho$ is the density matrix of the system, the reduced density matrix for the subsystem $A$ is

$$\rho_A = Tr_B[\rho] .$$

(8.35)

The entanglement entropy (EE) of the subsystem $A$, also known as the von-Neumann entropy, is defined as

$$S_{vN} = -Tr [\rho_A \log(\rho_A)] .$$

(8.36)

At zero temperature, the expression of $S_{vN}$ for a finite 1d system of length $L$ and divided into two pieces of length $x$ and $(L - x)$ is given by the Calabrese-Cardy (CC) formula [218]

$$S_{vN} = \frac{c}{6} \log \left( \frac{2L}{\pi} \sin \left( \frac{\pi x}{L} \right) \right) + \tilde{c}_1 ,$$

(8.37)

where, $c$ is the central charge of the underlying conformal field theory which measures the number of “degrees of freedom” (DoF) in the system and $\tilde{c}_1$ is a non-universal constant. We calculate the EE using the DMRG as a function of the subsystem size and then fit it with the CC formula to extract the central charge. We show the variation of the central charge $c$ in the whole $\phi - \Omega$ plane in fig. 8.9(c). We note that for large values of $\Omega$ in the SM phase since there is only one DoF left in the system, the central charge is $c = 1$. Similarly, in the phases DSF, SDSF and FSDSF also there is only one DoF left giving $c = 1$. Although the DoF in these phases is “different” from that of the SM phase. But, in the FFLO (F) phase $c$ is different from 1 as expected.

The transition from the SM to a quasi-superfluid phase, seen in fig. 8.9 for the “large” $\Omega$ limit as a function of $\phi$, can be understood by looking at the corresponding non-interacting system. The single particle dispersions of the two bands for the $M = 2$ case are given by

$$\epsilon_1(k) = -2t \cos \left( \frac{\phi}{2} \right) \cos \left( \frac{\phi}{2} - k \right) - \sqrt{\Omega^2 + 4t^2 \sin^2 \left( \frac{\phi}{2} \right) \sin^2 \left( \frac{\phi}{2} - k \right)} ,$$

(8.38)
8.6. PHASE DIAGRAM OF THE $M = 2$ CASE

Figure 8.10: (Color online) **Lifshitz transition**: The change in the topology of the Fermi surface of the SD system is shown for the non-interacting $M = 2$ case as a function of flux $\phi$ at “large” values of $\Omega$ for the density $n = 0.1$ and $t = 0.5$. In the left panel, we show the number of Fermi points in this regime of $\Omega$-$\phi$ plane. The right panels show the position of the chemical potential of the system for this filling and it is always in the first band ($\epsilon_1(k)$). We note that for $\phi = 0.2\pi$ (top right panel) there are 2 Fermi points whereas for $\phi = 0.7\pi$ (bottom right panel) there are 4 Fermi points.

$$\epsilon_2(k) = -2t \cos\left(\frac{\phi}{2}\right) \cos\left(\frac{\phi}{2} - k\right) + \sqrt{\Omega^2 + 4t^2 \sin^2\left(\frac{\phi}{2}\right) \sin^2\left(\frac{\phi}{2} - k\right)}.$$  \hspace{1cm} (8.39)

By filling particles with the density $n = 0.1$ corresponding to fig. 8.9, we show in fig. 8.10 that for the “large” $\Omega$ limit as a function of $\phi$, there is a change in the topology of the Fermi surface of the non-interacting $M = 2$ SD system. It changes from having 2 Fermi points to having 4 Fermi points by changing $\phi$ from 0 to $\pi$. Hence, the transition from the SM phase to a quasi-superfluid phase in the large $\Omega$ limit occurs due to this change in the topology of the Fermi surface just discussed.

The lowest $m$-particle excitation energy ($\Delta E_m$) above the ground state is defined as [219]

$$\Delta E_m = \left[\frac{E_0(N + m, L) + E_0(N - m, L) - 2E_0(N, L)}{m}\right],$$  \hspace{1cm} (8.40)

where, $E_0(N, L)$ is the ground state energy of a system with $N$ particles and $L$ physical sites. This definition can be related to the change in the chemical potential per particle in the system due to addition and removal of $m$ particles in the many body ground state. We analyze the single particle and two particle exci-
8.6. PHASE DIAGRAM OF THE M = 2 CASE

Figure 8.11: (Color online) Excitation energies for the M = 2 case with L = 80, n = 0.1 and t/U = 0.5. The panels (a), (b) and (c) show the variation of the single particle and two particle excitation energies as a function of Ω for different values of φ = 0.05π, 0.2π and π respectively. The panel (d) shows the variation of the same two quantities but as a function of flux φ for a large value of Ω/U = 0.8. The region within the two black dotted lines in panel (a) denotes the extent of the FFLO region and the black dotted line in (b) marks the point where the transition from the DSF to the SM phase occurs whereas that in (d) marks the point where the transition from the SM to the SDSF phase occurs.

tation energies for the M = 2 case as a function of Ω for different flux and are shown in the fig. 8.11. We note that as expected for small Ω, in fig. 8.11(a) and fig. 8.11(b), the single particle excitation energy is the lowest for a general flux (≠ π) due to the presence of the paired phase (DSF). But as Ω increases since the characters of different phases change, we also see the effect in these excitations. For very small flux φ = 0.05π, it is seen form fig. 8.11(a) that for very large Ω the one particle excitation energy is lowest and ΔE₂ = 2ΔE₁; signaling the formation of a single component metallic (SM) phase. Similar behavior is seen for large Ω for the intermediate flux case φ = 0.2π shown in fig. 8.11(b). But we note that
fig. 8.11(a) has some extra feature for in between $\Omega$ values due to the presence of the FFLO region. We also note one interesting aspect that for any values of $\Omega$ with the $\pi$ flux case, shown in fig. 8.11(c), the two particle excitation energy is always the lowest. Hence, for this case there is a paired quasi-superfluid phase for any value of $\Omega$, although the characters of the paired objects are different. For small $\Omega$, the paired objects are usual dimers, whereas for large $\Omega$ they are fully squished dimers. Finally, we show the excitation spectrum of the system for a “large” value of $\Omega$ as a function of $\phi$ in fig. 8.11(d). As expected, the single particle excitations are the lowest for small $\phi$ in the SM phase; whereas at large $\phi$ the two particle excitations are the lowest due to the presence of a quasi-superfluid phase.

8.7 Summary

To summarize, we have investigated the many body physics of the SD system in this chapter with a particular focus on looking at the consequences of the novel phenomena happening in the single particle and the few body sectors of the SD system. We use DMRG and field theoretic technique such as Bosonization to demonstrate novel many body physics of the system. We showed that FFLO states get stabilized even without “imbalance” which is in contrary to the formation of the FFLO states in the imbalanced Fermi gases. By moving to the flavor basis, which transparently demonstrates squishing of the baryons, we show that there are several parameter regimes of the system in which analytical solutions are possible. We also show that this system with $\pi$-flux and large $\Omega$ has an emergent $U(1) \times U(1)$ symmetry and an effective theory for particles moving only in the lowest leg can be constructed. This system becomes a Luther-Emery liquid with dominant pairing correlations. Then, we note that a superfluid state can be generated from a ferromagnetic state by applying a magnetic field in the large $\Omega$ limit. Finally, by looking at different properties of the many body ground states like the entanglement entropy, central charge, lowest energy excitation spectrum etc., we analyze the full many body phase diagram of the system taking the $M = 2$ case as an example.
Chapter 9

Outlook

In this thesis, we have investigated the physics of the ultracold fermions in different geometries and different dimensions. We have shown that ultracold many body fermionic systems give rise to novel quantum phases due to an interplay between interaction and synthetic gauge fields. From these results, we conclude that these systems provide an unique opportunity to emulate the physics of strongly interacting many body systems. Finally, in the following section we describe the promising future directions of the topics presented in the different chapters of this thesis.

9.1 Future directions

We mentioned in chapter 3 that there has been a renewed interest [128–130, 133, 134] in studying the physics of quasi-2d Fermi gases. It was shown that due to the change in the depth of the confining optical potential, a dimensional crossover is induced in a 3d spin-$\frac{1}{2}$ fermionic system interacting via a contact interaction. It will be very interesting, then, to construct an effective theory for the system in the deep optical potential limit which will provide deeper insights into the problem. Another important direction is to investigate the variation of the superfluid density tensor with the change in the potential depth to understand the generalized BKT transition occurring in the system. Also, the study of the dimensional crossover in the presence of synthetic non-Abelian gauge fields can be interesting.

We already discussed in chapter 4 that there are ample possibilities of realiz-
9.1. FUTURE DIRECTIONS

ing novel quantum Hamiltonians arising due to an interplay between the trapping potential and the synthetic non-Abelian gauge fields. These possibilities of realizing quantum phases like quantum Hall states and a laboratory realization of a magnetic monopole have the potential to stimulate further interests in this field.

The novel physics happening in the SD system described in the chapters: chapter 5, chapter 6, chapter 7 and chapter 8; suggest that this system is interesting in its own right and there is a large number of unexplored possibilities. For example, interesting quantum lattice models can be realized in this system. The random flux model [206] having interesting properties can be realized in the SD system by making the Raman couplings random. This model is otherwise inaccessible in conventional condensed matter systems. A multiflavor generalization of an interesting topological model, called the Creutz ladder model [220] can be realized in this system. The main ingredient necessary for this realization is to generate nearest neighbor spin-flip terms. These can be achieved by following the proposal of Mazza et. al. [221] to induce controlled Raman transitions between nearest neighbor states with different flavor particles.

Studying the localization physics in this system is an unexplored area. As a function of the number of hyperfine states (which can be very “large”), localization physics in the crossover from 1 to 2 dimensions can be explored. Also, the case of repulsive interaction as well as bosonic systems can be interesting. And, recent developments in realizing the SD system through lattice shaking [222] and in integrated photonics [223] open up a whole new possibilities.

Study of the dynamics in the interacting SD system is an unexplored area. The theoretical studies, so far, have been concentrated on investigating the static properties of the SD system. It will be interesting to investigate the dynamical properties of this interacting SD system. In fact, the experimental realizations of the non-interacting SD system have investigated the quenching of the tunneling along the synthetic direction (Ω) and showed interesting dynamical signatures on the physical properties of the system. Also, there has been an extensive recent interest in studying the quenching properties [224, 225] of the one dimensional Fermi Hubbard model. It was shown that an interaction quench in the one dimensional Fermi-Hubbard model leaves fingerprint signatures on the emergent
9.1. FUTURE DIRECTIONS

FFLO correlations present in this system. This system is actually equivalent to the SD system with two-component fermions with SU(2) symmetric interaction and zero flux. Hence, for the interacting SD system, we can explore two very interesting directions: i) quenching of $\Omega$ for a fixed interaction strength ($U$) and ii) quenching of $U$ for a fixed $\Omega$. Also, studying the quenching properties of the SD system from a “ferromagnetic” state to a “superfluid” state with the quenching of the flux at large $\Omega$ will be interesting.
References


REFERENCES


