Optical frequency metrology with an Rb-stabilized ring-cavity resonator—study of cavity-dispersion errors

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We have developed a technique to measure the absolute frequencies of optical transitions by using an evacuated Rb-stabilized ring-cavity resonator as a transfer cavity. The absolute frequency of the Rb D2 line (at 780 nm) used to stabilize the cavity is known and allows us to determine the absolute value of the unknown frequency. We study wavelength-dependent errors due to dispersion at the cavity mirrors by measuring the frequency of the same transition in the Cs D2 line (at 852 nm) at three cavity lengths. The spread in the values shows that dispersion errors are below 30 kHz, corresponding to a relative precision of 10−10. We give an explanation for reduced dispersion errors in the ring-cavity geometry by calculating errors due to the lateral shift and the phase shift at the mirrors, and show that they are roughly equal but occur with opposite signs. We have earlier shown that diffraction errors (due to Guoy phase) are negligible in the ring-cavity geometry compared to a linear cavity; the reduced dispersion error is another advantage. Our values are consistent with measurements of the same transition using the more expensive frequency-comb technique. Our simpler method is ideally suited for measuring hyperfine structure, fine structure, and isotope shifts, up to several hundreds of gigahertz. © 2012 Optical Society of America

1. INTRODUCTION

Precise measurement of the absolute frequencies of atomic transitions has played an important role in the development of physics. This is perhaps best exemplified by the measurement of the small Lamb shift between the 2S1/2 and 2P1/2 levels of hydrogen, which led to the birth of quantum electrodynamics (QED), our most-successful physical theory to date. Since the SI standard of frequency or time is defined using a hyperfine transition in the Cs atom (≈9.1 GHz, which is in the microwave region), it is a challenge to measure the absolute frequency of an optical transition for it means that one has to span 6 orders of magnitude without loss in accuracy. In recent times, the use of a femtosecond frequency comb (1,2) generated by a laser beam in a nonlinear fiber has enabled such a span. The uniform spacing of the comb in the microwave region and directly referenced to the atomic clock, while the laser frequency is in the visible region. The inventors of this technique, Hänsch and Hall, won the Nobel Prize for this work.

Before the invention of the comb technique, wavelength measurement using cavities, especially the linear Fabry–Perot kind, was well developed. Through careful control of systematic errors due to dispersion, such measurements could achieve an uncertainty of around 10−10. Important results such as a precision measurement of the Rydberg constant and the proton-to-electron mass ratio using this method have been reported (3,4). For about a decade, we have pioneered the use of a similar cavity-based technique, but one that uses a ring-cavity resonator instead (5–7). As we will see later, the use of a ring cavity has several advantages over the use of a linear cavity. Though our technique is not competitive with the comb technique in terms of relative uncertainty (10−10 for the cavity and versus 10−16 for the comb), our simpler and lower-cost alternative can give similar accuracy when the spectrometer or the measured transition itself limits the uncertainty. For example, electric-dipole-allowed transitions in the visible have a Q of about 105, so a fractional uncertainty of 10−10 requires splitting the line by 1 in 100, which is a considerable experimental challenge. Our technique uses an evacuated ring-cavity resonator whose length is calibrated against a hyperfine transition in the D2 line of Rb. The absolute frequency of this line has been measured to a relative accuracy of 1.4 × 10−11 using a frequency chain referenced to the atomic standard (8). Thus, the Rb line acts as a secondary standard in the optical regime and the ring cavity as a transfer cavity, so that the absolute frequency of an unknown laser can be determined. In addition, frequency differences used in determining hyperfine structure, fine structure, and isotope shifts are in the 1 MHz–100 GHz range. Our technique is ideally suited for measurements in this range because almost all the sources of systematic errors will cancel when taking the difference (9–11).

The basic idea of the measurement scheme can be understood by referring to Fig. 1. The cavity is brought into simultaneous resonance for both lasers, so that the ratio of the two frequencies is a ratio of two integer mode numbers. While the resonance condition cannot be satisfied simultaneously for two arbitrary frequencies, it can be satisfied if we use an acousto-optic modulator (AOM) to offset the frequency of one of the lasers (the reference laser in our case). The resonance condition for a cavity of length L can be written as
is the frequency. Thus, the line of $\nu$ is the mode number and is different at the reference and unknown wavelengths. We at the wavelengths transition) at $\phi$ in a ring cavity wavefront curvature at their locations, and this ensures that mirrors in the cavity impose a boundary condition on the optical frequency, Eq. (2) shows that the technique is essentially a wavelength comparison. We must therefore be careful about possible systematic errors due to wavelength-dependent effects. We have shown earlier that the Guoy phase (due to diffraction) is wavelength independent in a ring cavity and therefore does not constitute a source of error [7]. The Guoy phase is given by $\arctan(z/z_R)$, where $z$ is the propagation distance and $z_R = \pi w_0^2/\lambda$ is the Rayleigh range around a waist of radius $w_0$. In our cavity, the waist size is proportional to $\sqrt{\lambda/\pi}$; therefore, the Rayleigh range is identical for all wavelengths, and so is the Guoy phase. Basically, the concave mirrors in the cavity impose a boundary condition on the wavefront curvature at their locations, and this ensures that the Guoy phase is independent of wavelength.

Another potential error arises due to wavelength-dependent phase shifts (as a result of dispersion) at the cavity mirrors. This can modify the cavity-resonance condition in Eq. (1) either due to a phase offset or a lateral shift in position at the mirrors, or both. Modeling this as an additional phase shift of $\phi$ in one round trip in the cavity, the modified resonance condition becomes

$$2n\pi = \frac{2\pi}{\lambda} L + \phi.$$

This makes the measured frequency dependent on $L$ as

$$\nu(L) = \nu_0 + \frac{c\phi}{2\pi L},$$

where $\nu_0$ is the correct frequency. This can lead to a systematic error if $\phi$ is different at the reference and unknown wavelengths. However, we have a good experimental handle to address this source of error, namely to measure the frequency at different cavity lengths. Equation (4) shows that $\phi$ can be eliminated by measuring the frequency for at least two cavity lengths. Physically, since the phase error due to dispersion at the mirrors is constant while the accumulated phase due to light propagation increases with length, the values at different cavity lengths helps us reduce the error.

In this work, we present a detailed study of errors due to mirror dispersion by measuring the frequency of the D$_2$ line in Cs at three cavity lengths. In earlier work [7,10], we put a limit on this error by varying the cavity length, but we could vary it only by about 25%; here we vary it by almost a factor of 2. We find the standard deviation in the values over this variation in length to be less than 30 kHz ($\sim 10^{-10}$ relative uncertainty). We give an explanation for reduced dispersion error in the ring cavity (compared to a linear cavity) by comparing the errors due to the lateral shift and the phase shift at the mirrors. We find that the two errors are roughly equal in magnitude but occur with opposite signs, so that they cancel to a large extent. This is another advantage of the ring geometry since there is no lateral shift in a linear cavity. We also compare our values to measurements on the same transitions made by the Hänsch group using a frequency comb. The two sets have similar error bars and agree well with each other.

2. EXPERIMENTAL DETAILS

The experimental details are shown in Fig. 1. The reference laser is a home-built diode laser system that is frequency stabilized using grating feedback [12]. The linewidth of the free-running laser is about 1 MHz. It is locked to a hyperfine transition on the D$_2$ line of $^{87}$Rb ($S_{1/2} \rightarrow P_{3/2}$ transition) at 780 nm using frequency modulation in a saturated-absorption spectroscopy setup. The unknown laser is a similar diode laser system operating at 852 nm. It is similarly locked to a hyperfine transition on the D$_2$ line of $^{133}$Cs ($S_{1/2} \rightarrow P_{3/2}$ transition), but modulated at a slightly different frequency. The two laser beams with orthogonal linear polarizations are mixed on a polarizing beam splitter (PBS) and coupled into a single-mode fiber before entering the cavity. The use of the fiber guarantees that the two beams enter the cavity at exactly the same angle, thereby avoiding potential errors due to differential excitation of higher-order cavity modes.

The cavity is a ring cavity in a bow-tie arrangement, as shown in Fig. 1. It consists of four mirrors, two plane and two concave, with a $15^\circ$ angle of incidence on the mirrors. The concave mirrors have radius of curvature of 25 mm. One of the plane mirrors is mounted on a piezoelectric transducer (PZT) so that the cavity length can be tuned electronically. The maximum displacement of the PZT is 6.2 $\mu$m with an applied voltage of 150 V. All the mirrors have commercially available multilayer dielectric coatings. The two concave mirrors and the PZT-mounted plane mirror have high reflectivity (>99%) over the range of 760–900 nm. The first plane mirror is partially reflecting so that it can be used as an input coupler. It is designed to have a reflectivity of ~97% at the wavelengths.
needed for the measurement (in this case 780 nm and 852 nm). The mirrors are mounted on ultrastable kinematic mounts. The mounts are fixed to a copper plate that is temperature stabilized using a thermoelectric cooler to increase the passive stability of the cavity. The entire assembly is placed inside a vacuum chamber and maintained at a pressure of $\sim 10^{-3}$ torr.

The cavity is analyzed using the standard ABCD matrix for Gaussian-beam propagation [13]. It is necessary to analyze the sagittal and tangential planes separately because of the $15^\circ$ angle of incidence on the curved mirrors, which makes the cavity modes elliptical. The cavity has two beam waists, a larger one between the plane mirrors and a smaller one between the curved mirrors. The beam from the fiber is mode-matched into the larger waist using a 50 cm focal length lens. For stable operation, the direct distance between the concave mirrors has to be around 26 mm (approximately equal to the radius of curvature), while the distance through the plane mirrors can be varied from 150 to 300 mm.

The beams entering the cavity (coming out of the single-mode fiber) have a Gaussian profile with $1/e^2$ diameter of 2 mm and power of 1 mW each. The power reflected from the input coupler of the cavity is split into its two linear components using a PBS so that the signals from the two lasers can be separated. The cavity modes have a linewidth of about 25 MHz. The PZT is modulated at 35 kHz with sufficient depth of modulation to generate error signals for locking to a cavity resonance. The demodulated signal from the unknown laser is used to lock the cavity on a peak, while the demodulated signal from the reference laser is used to lock the frequency of the AOM driver. This frequency is read using a frequency counter with a timebase stability of $10^{-8}$ over the integration time of 10 s used in the experiment. The AOM has a frequency range of 30 MHz centered around 100 MHz.

The procedure for establishing the cavity length uniquely has been described in detail in our earlier work [7]. It basically uses two steps: the first is to get a coarse measurement of the unknown frequency using a home-built wavemeter [12] (with an accuracy of $\sim 20$ MHz), and the second is to measure the cavity free spectral range (FSR). The FSR is determined by measuring the same unknown frequency with the reference laser on an $F = 1 \rightarrow F'$ transition and on an $F = 2 \rightarrow F'$ transition. This changes the frequency of the reference laser by a known amount of nearly 6.5 GHz [14]. Then there is only one mode number combination that satisfies Eq. (2), and the coarse frequency of the unknown laser, which gives the FSR and the length.

### 3. RESULTS AND DISCUSSION

The results of measurements at three cavity lengths—177, 220, and 315 mm—are listed in Table 1. There are three hyperfine transitions starting from the upper level of the ground state, and we have measured the frequency for each transition. Each transition is also measured with the reference laser both on an $F = 1 \rightarrow F'$ transition and an $F = 2 \rightarrow F'$ transition. The values listed are determined as follows. First, a given transition is measured for about 30 independent times with an integration time in the frequency counter of 10 s. The standard deviation of the set is about 15 kHz. Then the measurement is repeated on another day with complete resetting of the parameters of the various lock-in amplifiers. Again, the set of 30 independent readings shows a standard deviation of about 15 kHz. We verify that the two sets are consistent within their errors and take an average of all the readings. Even though the statistical error in the average is less than 10 kHz, we estimate the total error in each value to be 30 kHz due to the linewidth of the cavity modes.

The important thing to note from the table is that all six values for a given transition are consistent with each other within their error bars, and the standard deviation of each set is less than 30 kHz. To see this graphically, we show the measured frequencies of the $4 \rightarrow 3$ transition versus cavity length in Fig. 2. The straight line is a weighted average and the measurements at the two reference frequencies are randomly distributed around it. The weighted average has an error of 12 kHz. The other transitions show similarly consistent variation and have an error in the weighted average of 12 kHz. Thus, there is no significant change in the values as we increase the cavity length, showing that errors due to dispersion at the mirrors are less than 30 kHz, representing a relative precision of $10^{-10}$. This is consistent with the limit on this error from earlier studies [7,10], but there we were only able to reduce the length from 220 nm to 177 nm. Here, we find a similar limit even when we increase the length to 315 mm.

It therefore seems likely that there is some effect that causes the wavelength dependence to reduce in our ring-cavity geometry. As a potential explanation, consider the phase shift experienced by a light field when it reflects from a multilayer dielectric mirror. The dielectric layers are modeled as a transmission line, as detailed in Appendix A. The

### Table 1. Transition Frequencies Measured at Three Cavity Lengths

<table>
<thead>
<tr>
<th>Cavity Length (mm)</th>
<th>Ref. Laser Lock Point</th>
<th>Frequency for $F \rightarrow F'$ (MHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>177</td>
<td>4 $\rightarrow$ 3</td>
<td>508.241(30) 709.531(30) 960.326(30)</td>
</tr>
<tr>
<td></td>
<td>4 $\rightarrow$ 4</td>
<td>508.232(30) 709.538(30) 960.322(30)</td>
</tr>
<tr>
<td></td>
<td>4 $\rightarrow$ 5</td>
<td>508.234(30) 709.514(30) 960.398(30)</td>
</tr>
<tr>
<td>220</td>
<td>1 $\rightarrow$ F'</td>
<td>508.265(30) 709.560(30) 960.391(30)</td>
</tr>
<tr>
<td></td>
<td>2 $\rightarrow$ F'</td>
<td>508.236(30) 709.545(30) 960.362(30)</td>
</tr>
<tr>
<td>315</td>
<td>1 $\rightarrow$ F'</td>
<td>508.265(30) 709.560(30) 960.391(30)</td>
</tr>
<tr>
<td></td>
<td>2 $\rightarrow$ F'</td>
<td>508.236(30) 709.545(30) 960.362(30)</td>
</tr>
</tbody>
</table>

*There are two values at each length, measured with the reference laser locked to either an $F = 1 \rightarrow F'$ or an $F = 2 \rightarrow F'$ transition. For convenience, a constant offset of 351,721,000 MHz has been removed from all values.*

![Fig. 2. (Color online) Measured frequencies of the $4 \rightarrow 3$ transition versus the cavity length. The solid circles are measurements with an $F = 1 \rightarrow F'$ transition, and the open circles are with an $F = 2 \rightarrow F'$ transition. The line represents the weighted average yielding a value of 241(12) kHz.](image-url)
calculated reflectance and phase shift (for both TE and TM polarizations) for a typical mirror at 15° angle of incidence are shown in Fig. 2. The layer specifications are obtained from one of our regular mirror suppliers (Casimir’s Sieniutz Personal Enterprises, Lithuania), namely 29 layers of alternating refractive indices of $n = 1.45$ and $n = 2$, each layer having a thickness of $\lambda/4$ at the center wavelength of 800 nm. The variation with wavelength arises mainly due to changes in the optical path length, while wavelength-dependent changes in the refractive index play a negligible role. We verify this by doing calculations with and without the changes in the refractive indices given by the Cauchy dispersion formula,

$$n(\lambda) = A + \frac{B}{\lambda^2} + \ldots$$  \hspace{1cm} (5)$$

using typical values for the $A$ and $B$ coefficients for the two kinds of layers. Hence the results in Fig. 3 are shown assuming wavelength-independent refractive indices. As expected, the calculated reflectance is close to 100%, which serves as a consistency check on our calculation procedure.

Now, let us consider the effect of this phase shift on the measurement. If the shift were perfectly linear with frequency, it would not cause an error in our measurement. To see this, consider that the total phase shift varies as $A\nu$. Then the resonance condition after one round trip becomes [see Eq. (3)]

$$2n\pi = \frac{2\pi\nu}{c}L + A\nu = \frac{2\pi}{c}(L + Ac)\nu.$$  \hspace{1cm} (6)$$

This means that the determination of the unknown frequency from Eq. (2) will remain unaffected. In effect, the linear part of the phase shift acts like a change in the overall cavity length from $L$ to $L + Ac$, and causes no error in the measurement. Therefore, we only have to consider the deviation from linear in our error analysis. The deviation between 780 nm and 852 nm is about 10 nmrad, or about 100 times smaller than the size of the shift.

Next, consider the lateral shift in the reflected ray at a multilayer dielectric mirror, shown in Fig. 4(a) after reflection from one set of layers. The distance between the point of emergence and the point of incidence of the ray is defined as the lateral shift. One can think of this as arising due to the ray penetrating the mirror surface and traveling some distance in the mirror. Therefore, for the same distance, the phase of the emerging ray will increase linearly with frequency. This means that there will be no change in the lateral shift if there is a linear dependence of the phase shift. The part that will cause an error is the deviation from linear, the same as for the phase shift calculated above.

The error in the frequency measurement due to the lateral shift can be determined by tracing the path of a ray in the ring cavity. The ray paths for two different lateral shifts are shown in Fig. 4(b). Similar ray tracing, assuming a lateral shift of $x$ at one of the mirrors and zero at the others, shows that the total free-space length in the cavity will decrease by $3x\sin \theta_i$. This means that an increase in the lateral shift (due to an increase in the phase shift) will result in a decrease in the path length. Thus the two effects, phase shift and lateral shift, will act in opposite directions and cancel to some extent.

The above calculation is done with plane waves and does not give a direct estimate of the lateral shift. However, we can get a rough estimate by taking the phase shift of the emerging ray as arising due to the lateral shift; i.e., the phase at the point of emergence in Fig. 4(a) is used to calculate the lateral shift. The results of such a calculation are shown in Fig. 5. As expected, the linear phase shift gives a constant shift of about 154 nm, while the deviation from linear (shown in the figure) results in a variation of about 1% of the total shift. The change in lateral shift from 780 nm to 852 nm is 1 nm. Using the path difference of $3x\sin \theta_i$ for a round trip in the cavity, this leads to a decrease in phase of 0.3 nmrad, which is of the same order of magnitude as the error due to the phase shift but opposite in sign.

Even though our calculation is a model calculation and does not take into account the curvature of the mirrors, it shows that
the errors due to the phase shift and lateral shift are nearly equal and opposite. This is the likely reason for the reduced dispersion effects in the ring-cavity geometry. By contrast, a linear cavity would show a similar phase shift at the mirror (which we have verified in our calculations using 0° angle of incidence), but would not have any lateral shift. Thus, reduced dispersion error is another advantage, over the absence of diffraction error, of the ring geometry for such measurements.

In the above set of measurements, we did not have to worry about spectroscopic errors since we were measuring the same transition at the three lengths. Any errors in our Rb or Cs spectrometers would be the same at all three lengths, and would therefore cancel in estimating the dispersion error. In effect, we were only interested in finding out if the measured value changes, and not the value itself. However, if we can estimate the error in the two spectrometers, we can make a direct comparison of our values to measurements on this line using the frequency-comb technique made by the Hänsch group [15]. The potential sources of error in our spectrometers are listed below.

1. Systematic Shifts of the Rb (Reference) Laser
   (i) Shift in the peak position can occur due to (a) optical-pumping effects and (b) velocity redistribution of the atoms in the vapor cell due to radiation pressure [16]. Such effects manifest themselves as inversion of hyperfine peaks or distortion of the Lorentzian lineshape. We minimize these effects by using very low intensities of the pump and probe beams in the saturated-absorption spectrometer. The typical intensity in the probe beam is about 0.25 mW/cm² compared to the saturation intensity of 1.64 mW/cm². The pump beam has an intensity three times higher.

   (ii) Line shifts from stray magnetic fields in the vicinity of the cells. The primary effect of a magnetic field is to split the Zeeman sublevels and broaden the line without affecting the line center. However, line shifts can occur if there is asymmetric optical pumping into Zeeman sublevels. For a transition \((F', m_F) \rightarrow (F, m_F')\), the systematic shift of the line center is

   \[
   \Delta m = \mu_B (g_F m_F - g_m m_F) B,
   \]

   where \(\mu_B = 1.4\ \text{MHz/G}\) is the Bohr magneton, \(g_s\) denote the Landé g factors of the two levels, and \(B\) is the magnetic field. The selection rule for dipole transitions is \(\Delta m = 0, \pm 1\) depending on the direction of the magnetic field (quantization axis) and the polarization of the light. Thus, if the beams are perfectly linearly polarized, there will be no asymmetric driving and the line center is not affected. We therefore use linearly polarized light to reduce these effects. We further verify that the effects are negligible by repeating the measurements at different locations in the laboratory, and with and without the use of a magnetic shield around the cell.

   (iii) Shift in the lock point due to the underlying Doppler profile or peak-pulling from nearby transitions. We minimize these effects by using third-harmonic detection for the error signal.

2. Systematic Shifts of the Cs (Unknown) Laser
   (i) For the Cs D₂ line, the effects of radiation pressure are similarly reduced by using a low intensity in the probe beam of 0.20 mW/cm², which is again much smaller than the saturation intensity of 0.83 mW/cm². The most important effects come from optical pumping and stray magnetic fields, which are again reduced by using linear polarization.

   The estimated values for the various sources of error, given in Table 2, are justified in our earlier work [7]. For example, the error due to optical pumping and radiation pressure is estimated by studying the asymmetry in the spectral lineshape at different intensities. The error due to stray fields is the maximum shift by assuming that the atoms get optically pumped into the stretch state in the presence of a stray field of 10 mG. It varies from 1 to 14 kHz for different transitions, and is taken at the maximum value for all transitions. The error due to the uncertainty in the cavity lock (determined by the width of the cavity mode) is listed under the two lasers but is really applicable to the error in locking the cavity and the AOM. Though this error could be less for the average at the three lengths, we have left it at the value for a single measurement because there could be a common systematic error that does not average out. The total uncertainty obtained by adding the various sources in quadrature is 50 kHz.

   The average value for each transition with this error estimate is compared to previous frequency-comb measurements by the Hänsch group in Table 3. Their error is similar to ours and shows that the spectrometer limits the measurement rather than the technique of frequency measurement. The values overlap for all three transitions within 1σ of the combined error, showing that there is no unknown systematic error plaguing our spectrometer.

   The measured values in Table 1 show that errors due to dispersion are under control when we change the cavity length by almost a factor of 2, showing that the measurement technique is reliable at the 30 kHz level of precision. We have given an explanation for this reduced wavelength dependence in terms of opposite signs of the error due to the phase shift and the lateral shift at the cavity mirrors. We have already shown that one of the advantages of a ring cavity over a linear cavity is the absence of diffraction effects (Guoy phase) [7]. The reduced dispersion error is another advantage of the ring geometry because this will not happen in a linear cavity where the lateral shift is absent (because of the zero angle of

   ![Fig. 5. (Color online) Calculated deviation in lateral shift as a function of frequency (for TE polarization).](image)

<table>
<thead>
<tr>
<th>Table 2. Error Budget</th>
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<tbody>
<tr>
<td>Source of Error</td>
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<tr>
<td>Ref. Laser Cs D₂</td>
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<tr>
<td>Optical pumping and radiation pressure</td>
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<tr>
<td>Stray magnetic fields</td>
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<tr>
<td>Laser lock to spectral peak</td>
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<td>Locking to cavity resonance</td>
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incidence on the mirrors). Thus, the Rb-stabilized ring cavity appears to be a useful and simpler alternative to the frequency-comb method for measurements in the relative precision range of $10^{-10}$. In addition, linear cavities are routinely used to measure fine/hyperfine structure and isotope shifts [17]; the use of a ring cavity instead may be advantageous for these applications. For example, we have recently used this cavity to make precise measurements of hyperfine structure in the $^3P_2$ state of $^{171}$Yb, and observe the nuclear magnetic octupole constant for the first time in this class of atoms [18].

APPENDIX A: TRANSMISSION LINE MODEL FOR MULTILAYER DIELECTRIC STRUCTURES

In Fig. 6(a), we show a general multilayer structure with $m$ nonmagnetic ($\mu = \mu_0$) dielectric layers forming a mirror. The last layer is the glass substrate. We need to find the reflection due to this structure when light is incident from the lower half space of air. For simplicity, we assume that all the interfaces are parallel to each other and perpendicular to the $z$ axis. The $j$th dielectric layer has a thickness $t_j$ and refractive index $n_j$. The calculations are done for an arbitrary angle of incidence, and both TE and TM polarizations.

Our approach is to make an equivalent transmission line structure for the given multilayer structure. To understand the equivalence, we consider the vectors $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{k}$ as shown in Fig. 6(b). For the $j$th nonmagnetic dielectric layer of refractive index $n_j$, the magnitude of the $\mathbf{k}$ vector is given by

$$k_j = n_j k_0,$$

where $k_0 = 2\pi /\lambda_0$ is the magnitude of the wavevector in free space. When treated as a transmission line segment, the propagation constant in this layer is the component of the $\mathbf{k}$ vector along the $z$ direction, so that

$$k_{z,j} = k_j \cos \theta_j = n_j k_0 \cos \theta_j.$$  

(A2)

In the layer, $\theta_j$ is related to $\theta_0$, the angle of incidence in air, by Snell's law:

$$n_j \sin \theta_j = n_0 \sin \theta_0.$$  

(A3)

Therefore, the characteristic impedance of the equivalent transmission line for the layer is given by

$$Z_{j,TE} = \frac{n_0}{n} \cos \theta_j$$

and

$$Z_{j,TM} = \frac{n_0 \cos \theta_j}{n},$$

(A4)

where $\theta_0 = \sqrt{\mu_0 /\varepsilon_0} \cdot 376.73$ $\Omega$ is the impedance of free space. Once the equivalent transmission line structure is constructed, calculations can be done in a variety of ways. We use an approach involving ABCD matrices.

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